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***In Memorian**

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ON THE OPERATOR-VALUED FOURIER TRANSFORM OF HARISH-CHANDRA SCWARTZ ALGEBRAS

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Abstract

We establish a $K-$type decomposition of the Harish-Chandra Schwartz algebra $\mathcal{C}^p(G)$, for any real-rank 1 reductive group G with a maximal compact subgroup K and $0 < p \leq 2$. This decomposition is then used to give an infinite-matrix-realization of the operator-valued Fourier image

$$\mathfrak{F} : \mathcal{C}^p(G) \to \mathcal{C}^p(\hat{G})$$

of $\mathcal{C}^p(G)$ as a Fréchet multiplication algebra in which every member of $\mathcal{C}^p(\hat{G})$ consists of a countable block-matrices of the form

$$((\mathfrak{F}_B(\breve{\alpha})_{(\gamma,m)}(\Lambda) \otimes \mathfrak{F}_H(\breve{\alpha})_{(\gamma,l)}(Q : \chi : \nu))_{\gamma \in F,(l,m)\in\mathbb{Z}^2})_{F\subset\hat{K},|F|<\infty}$$

for every $\alpha \in \mathcal{C}^p(G)$. This proves Trombi's conjecture for G of real rank 1 and the technique leads to a proof of the fundamental theorem of harmonic analysis for any arbitrary real-rank reductive group G.

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§1. **Preliminaries.**

This paper considers the operator-valued Fourier transform image of the L^p- version of *Harish-Chandra Schwartz* algebra for any real reductive group G in the *Harish-Chandra class.* To this end we introduce a method of reduction of the operator-valued problem to the consideration of only the scalar entries of a countably-infinite blocks of a block of matrices. That is, a reduction to the scalar entries of a countably-infinite matrix consisting of countably-infinite blocks of matrices. This perspective was motivated by Varadarajan [29.] in the presence of some Lemmas on the decomposition of the (necessarily) countable [32.] set $\hat{K}$ of equivalence classes of (necessarily finite) irreducible unitary representations of a maximal compact subgroup K of G.

This feat has been successfully employed for $G = SL(2,\mathbb{R})$ by both Barker [5.] for $0 < p \leq 2$ as a *Fréchet* space of linear operators and by Varadarajan [29.] for $p = 2$ as a *Fréchet* algebra of infinite matrices. It is a basic elementary exercise that every linear operator may be easily realized as a matrix consisting of finite or infinite entries. The methods of Barker and Varadarajan, which may together be termed *the type-(m,n) approach,* considered K as

$$K = \{u_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in (0, 2\pi]\} = SO(2)$$

and the finite set $\{m,n\}$ in $\hat{K} = \mathbb{Z}$, using them to discuss (in some details) subjects like the type-(m,n) spherical functions on $SL(2,\mathbb{R})$, asymptotic growth of matrix coefficients of the irreducible unitary representations of type-(m,n), (constructed) the orthogonal Schwartz-type Frĕchet algebras ${}^{o}\mathcal{C}^p_{mn}(SL(2,\mathbb{R}))$ and ${}^{\#}\mathcal{C}^p_{mn}(SL(2,\mathbb{R}))$ (where $\mathcal{C}^p_{mn}(SL(2,\mathbb{R}))$ consists of type-(m,n) functions α satisfying $\sup_{x\in G} |\alpha(x:a)|\Xi^{-2/p}(x)(1+\sigma(x))^r)$, computed their Fourier transforms, which were put together to arrive at the operator-valued Fourier transform of the entire $\mathcal{C}^p_{mn}(SL(2,\mathbb{R}))$, for $0 < p \leq 2$.

In its details, Varadarajan [29.] used an L^2-orthogonal basis $\{e_n\}_{n\in\mathbb{Z}}$ of the representation space of the regular representation π such that

$$\pi(u_\theta)e_n = e^{in\theta}e_n$$

and which splits into $\pi = (\pi_k, \pi_{\epsilon,i\mu})$: the discrete series $\pi_{k\in\mathbb{Z}}$ and the irreducible unitary principal series $\pi_{\epsilon,i\mu\in\{0,1\}\times\mathbb{R}}$. The type-$(m,n)$ subspace $L^2_{mn}(G)$

of $L^2(G)$ satisfying the requirement

$$f(u_{\theta_1} x u_{\theta_2}) = e^{i(m+n)\theta} f(x),$$

$x \in G$, was considered (See Ehrenpreis and Mautner [10.], [11.] and [12.]), a spanning example of which is the combined subspace of the subspaces of the matrix coefficients of π_k and $\pi_{\epsilon,i\mu}$ with respect to $\{e_n\}$ given respectively as

$$\phi_{kmn}(x) =< \pi_k(x)e_m, e_n >$$

and

$$f_{mn}(\mu : x) =< \pi_{\epsilon,i\mu}(x)e_m, e_n > .$$

Analysis of the perturbed differential equations satisfied by both types of matrix coefficients leads to the understanding of their asymptotic growths. In particular, we have that

$$e^t f_{mn}(\mu : u : a_t : v) \sim p^+(\mu : m : n)c^+(\mu)e^{i\mu t} + p^-(\mu : m : n)c^-(\mu)e^{-i\mu t},$$

where $\sim$ means the difference of the two sides is $O(e^{-2t})$; $p^{\pm}$ are polynomials in μ, m, n; $u, v \in \mathfrak{U}(sL(2,\mathbb{R}))$ (the universal enveloping algebra of the Lie algebra $\mathfrak{g} = sl(2,\mathbb{R})$ of $G = SL(2,\mathbb{R})$) and $c^{\pm}$ are the $SL(2,\mathbb{R})$ c–functions. The knowledge of this control of the growth of both ϕ_{kmn} and $f_{mn}(\mu : \cdot)$ suggests the consideration of the Harish-Chandra operator-valued Fourier transform on the Schwartz-type subspace $\mathcal{C}^2_{mn}(SL(2,\mathbb{R}))$ of $L^2_{mn}(SL(2,\mathbb{R}))$ as the map

$$f \mapsto \mathfrak{F}(f) =: (\pi_k(f), \pi_{\epsilon,i\mu}(f))$$

denoted as

$$((\mathfrak{F}_B f)(k), (\mathfrak{F}_L f)(\mu))$$

and given as

$$(\mathfrak{F}_B f)(k) := \int_G f\bar{\phi}_{kmn} dG$$

and

$$(\mathfrak{F}_L f)(\mu) := \int_G f\bar{f}_{mn}(\mu : \cdot) dG.$$

All these form the scalar-reduction of the theory which involves proving the continuous injection of $f \mapsto \mathfrak{F}_B f$ and $f \mapsto \mathfrak{F}_L f$ on the direct sum

$$^o\mathcal{C}^2_{mn}(G) \oplus^* \mathcal{C}^2_{mn}(G)$$

(as contained in [29.], Theorem 20 and first part of Theorem 57) and of the exact wave-packet (as contained in [29.], Theorem 33).

At the end of the long journey on the explicit asymptotic growth estimates, the direct sum

$$\mathfrak{F}_B(\mathcal{C}^2(SL(2,\mathbb{R}))) \oplus \mathfrak{F}_L(\mathcal{C}^2(SL(2,\mathbb{R})))$$

was found to be the (infinite) matrix realization of the operator-valued Fourier image of $\mathcal{C}^2(SL(2,\mathbb{R}))$ (c.f. Theorem 60 of [29.]). A short *coda* on invariant transform theory on $SL(2,\mathbb{R})$ completes the journey, leading to the following characterization.

1.1 Proposition. (Varadarajan [29.]) If T is a tempered distribution on $G = SL(2,\mathbb{R})$, then T is invariant iff T vanishes on all commutators, i.e., $T(f * g) = T(g * f)$, for all $f, g, \in \mathcal{C}^2(G)$. In this case, there are unique tempered distribution $\hat{T}_B$ on $\hat{B}$ and $\hat{T}_H$ on $\hat{H}$, with $\hat{T}_H$ symmetric, such that $T(f) = \hat{T}_B(f_B) + \hat{T}_H(f_H)$. □

Even though it was never claimed in the work of Varadarajan that the theory being developed in [29.] was based on the *'finite' $K-$type* results of Trombi [25.], [26.], it is however very clear that the parameters m and n (in $\mathbb{Z} \simeq \hat{K} = \hat{SO}(2)$) are the $K-$types of members of $\mathcal{C}^2_{mn}(SL(2,\mathbb{R}))$ and that the set $\{m,n\}$ was to be considered as a finite subset of the (necessarily countable) set $\hat{K} = \hat{SO}(2)$. Hence $F := \{m,n\}$ is a (representative) finite $K(= SO(2))-$type of $G = SL(2,\mathbb{R})$ and that each $\mathcal{C}^p_{mn}(SL(2,\mathbb{R}))$ (in the notation of the $L^2(SL(2,\mathbb{R}))$ results of Varadarajan [29.] and the $L^p(SL(2,\mathbb{R}))$ results of Barker [5.]) may simply be written as

$$\mathcal{C}^p_{mn}(SL(2,\mathbb{R})) = \mathcal{C}^p(SL(2,\mathbb{R}) : \{m,n\}) = \mathcal{C}^p(SL(2,\mathbb{R}) : F)_{|_{F=\{m,n\}}},$$

in the notation $\mathcal{C}^p(G : F)$ of Trombi [25.].

In order to then complete his L^p results on $SL(2,\mathbb{R})$, Barker [5.] appealed to an argument of Arthur which simply says that:

(*i*) the algebraic (direct) sum of all $\mathcal{C}^p_{mn}(SL(2,\mathbb{R}))$ is exactly the subspace $\mathcal{C}^p(SL(2,\mathbb{R}), SO(2))$ of $SO(2)-$finite members of $\mathcal{C}^p(SL(2,\mathbb{R}))$ and

(*ii*) the space $\mathcal{C}^p(SL(2,\mathbb{R}), SO(2))$ is dense in $\mathcal{C}^p(SL(2,\mathbb{R}))$ (See also Warner [32.], Proposition 4.4.3.5 on p. 256).

With these facts at hand (especially the denseness result in (*ii*) above, which is valid for $\mathcal{C}^p(G)$, for any real reductive group G), Barker only needed to

prove his isomorphism theorem for $\mathcal{C}^p(SL(2,\mathbb{R}),SO(2))$, as contained in [5.], Theorem 18.1, p. 97, where $\mathcal{C}^p(SL(2,\mathbb{R}),SO(2))$ is the space of $K-$finite members of $\mathcal{C}^p(SL(2,\mathbb{R}))$.

It then follows therefore that without any argument of Arthur [3.], as used in Barker [5.] to remove Trombi's finite $K-$type restriction (See [5.], p. 1), we could easily use the scalar results of $\mathcal{C}^p_{mn}(SL(2,\mathbb{R}))$ in Barker [5.] now seen as $\mathcal{C}^p(SL(2,\mathbb{R}):F)_{|_{F=\{m,n\}}}$ and is the subspace

$$\mathcal{C}^p(SL(2,\mathbb{R}):F)_{|_{F=\{m,n\}}} = \mathcal{C}^p(SL(2,\mathbb{R}):\{m,n\}) = \mathcal{C}^p_{mn}(SL(2,\mathbb{R}))$$

(in the notation of Varadarajan[29.]) to get matrix-realization of members of all $\mathfrak{F}(\mathcal{C}^p(SL(2,\mathbb{R})))$ in the style of Varadarajan [29.].

The question we set out to answer in this paper is: In what context could Trombi's space $\mathcal{C}^p(G:F)$ be seen as the space of type-F $(=\{m,n\})$ spherical function in $\mathcal{C}^p(G)$, so that their algebraic direct sum (which is already dense in $\mathcal{C}^p(G)$, being the subspaces $\mathcal{C}^p(G:F)$) could be deemed sufficiently dense for us to state te isomorphism theorem for the operator-valued Fourier transform of $\mathcal{C}^p(G)$. An answer to this question is contained in §4.

The present paper is organised as follows. Basic results on the subject (See [25.]) are summarised below while a short historical journey to the heart of the problem (in much technical details than the present introduction) is contained in §2. Our main results are in §3. (where the crucial Lemmas are proved), §4. (where we undertake the required $K-$type decompositions and establish Trombi's conjecture for all real-rank 1 groups G) and §5. (where we establish the fundamental theorem of harmonic analysis for all real reductive groups). A *coda* gives another perspective via $K-$finite functions in §6.

Let us give a first detail to our outlook. An introduction to the theory is contained in [17.]. Let G denote any real reductive group in the Harish-Chandra class (Varadarajan, [30.]) with Lie algebra $\mathfrak{g}$ and with a maximal compact subgroup K (in which $rk(G) \geq rk(K)$) so that, whenever $rk(G) = rk(K)$, we choose B for a Cartan subgroup of G contained in K. θ is a Cartan involution of G fixing K element-wise. If we also denote by $\mathfrak{b}$ the Lie algebra of B then there exists a lattice $L_B \subset \mathfrak{b}^*_{\mathbb{C}}$ such that L_B is isomorphic to the dual $\hat{B}$. Let $\mathfrak{h}$ denote a θ stable Cartan subalgebra of $\mathfrak{g}$ with maximal vector part so that

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{s}$$

is the corresponding Cartan decomposition of $\mathfrak{g}$. Set the non-empty set M to be cenralizer$_K(\mathfrak{h} \cap \mathfrak{s})$ while M' denotes normalizer$_K(\mathfrak{h} \cap \mathfrak{s})$. Denote $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{s}$.

The set $P(\mathfrak{a})$ (respectively, $P(A)$ with $A := \exp \mathfrak{a}$) shall denote the set of all those parabolic subalgebras (respectively, parabolic subgroups) whose split rank is $\mathfrak{a}$ (respectively, A).

The (finite) Weyl group $W(A) := M'/M$ acts on $\mathfrak{a}_{\mathbb{C}}^*$ and on $\hat{M}$ as

$$(\mathfrak{w}v)(H) = v(Ad\, \mathfrak{w}^{-1}(H)), \quad H \in \mathfrak{a}_{\mathbb{C}}^*$$

and

$$(\mathfrak{w}\sigma)(m) = \sigma(\mathfrak{w}^{-1}\, m\, \mathfrak{w}), \quad m \in M,$$

respectively, with $\mathfrak{w} \in W(A)$. Now let $Q = MAN_Q$ denote the Langlands decomposition of $Q \in P(A)$. For any $\chi \in \hat{M}$ choose $\sigma \in \chi$ and with $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ define ξ_ν on A as

$$\xi_\nu(a) = e^{\nu(\log a)}$$

and it follows that $\sigma \otimes \xi_\nu$ (defined on MA and) given as

$$(\sigma \otimes \xi_\nu)(ma) = \sigma(m)\xi_\nu(a)$$

can be extended to all of Q by making it unit on N_Q. The principal series of representations of G is given therefore as

$$\pi_{Q,\chi,\nu} = Ind_Q^G(\sigma \otimes \xi_\nu).$$

Consider any $Q \in P(A)$ and for the set $\hat{K}$ consisting of the set of equivalence classes of irreducible unitary (necessarily finite-dimensional) representations of K, we choose $F \subset \hat{K}$ whose cardinality $|F| < \infty$. We shall denote by $C_c^\infty(G : F)$ the space of compactly-supported smooth functions on G whose $K-$types are in F. That is, $\alpha \in C_c^\infty(G)$ satisfying the requirement that

$$d(\xi) \int_K \bar{\chi}_\xi(k)\alpha(\tau_l(k) \cdot x)dk = \alpha(x) = d(\xi) \int_K \alpha(x \cdot \tau_r(k))\bar{\chi}_\xi(k)dk$$

for all $\xi \in F$, where τ_l and τ_r denote the left and right actions of K on G and χ_ξ is the character of ξ. Equivalently, $\alpha \in C_c^\infty(G : F)$ if and only if

$$d(\xi)^2 \int_K \int_K \bar{\chi}_\xi(k_1)\alpha(\tau_l(k_1) \cdot x \cdot \tau_r(k_2))\bar{\chi}_\xi(k_2))dk_1dk_2 = \alpha(x).$$

A relation on $\hat{K}$ may be defined as follows: $\xi_1 \sim \xi_2$ whenever there exists $\alpha \in C_c^\infty(G : F)$ such that

$$d(\xi_1)d(\xi_2) \int_K \int_K \bar{\chi}_{\xi_1}(k_1)\alpha(\tau_l(k_1) \cdot x \cdot \tau_r(k_2))\bar{\chi}_{\xi_2}(k_2))dk_1dk_2 = \alpha(x).$$

This is an equivalence relation on $\hat{K}$, whose partitioning of $\hat{K}$ would be needed later. We may set $F(\xi_{j^*}) := \{\xi_j \in \hat{K} : \xi_{j_1} \sim \xi_{j^*}\}$. Clearly

$$\hat{K} = \bigcup_{\xi_{j^*}} F(\xi_{j^*})$$

is a disjoint union.

The Fourier transform of any $\alpha \in C_c^\infty(G : F)$ can be defined relatively to both the discrete and principal series of representations, π_Λ (where $\Lambda \in L_B^+$) and $\pi_{Q,\chi,\nu}$ (where $Q \in P(A)$, $\chi \in \hat{M}$ and $\nu \in \mathfrak{a}_{\mathbb{C}}^*$), respectively. Indeed, if we denote by

$$\alpha \mapsto \mathfrak{F}(\alpha)$$

the operator-valued Fourier transform of $\alpha \in C_c^\infty(G : F)$, then

$$\mathfrak{F}(\alpha) = \begin{cases} (0, \mathfrak{F}_H(\alpha)), & \text{if } rk(G) > rk(K), \\ (\mathfrak{F}_B(\alpha), \mathfrak{F}_H(\alpha)), & \text{if } rk(G) = rk(K), \end{cases}$$

where $\mathfrak{F}_B(\alpha)(\Lambda) := \pi_\Lambda^F(\check{\alpha})$ and $\mathfrak{F}_H(\alpha)(\Lambda) := \pi_{Q,\chi,\nu}^F(\check{\alpha})$. Here we define the map $\check{\alpha}$ as $\check{\alpha}(x) = \alpha(x^{-1})$ and we recall the general averaging-operator on G as

$$\pi(\alpha) = \int_G \alpha(x)\pi(x)dx.$$

Simply put,

$$\mathfrak{F}(\alpha) = (\mathfrak{F}_B(\alpha), \mathfrak{F}_H(\alpha))$$

(for all $\alpha \in C_c^\infty(G : F)$) where $\mathfrak{F}_B(\alpha) \equiv 0$ whenever $rk(G) > rk(K)$. A crucial lemma on the asymptotic growth of $\alpha \to \mathfrak{F}(\alpha)$ may be deduced as follows from that of $\alpha \to \mathfrak{F}_H(\alpha)$.

1.2 Lemma. (Trombi [25.], p. 102) Let $s > 0$. Then given any $N \geq 0$ there exists a constant C_N such that for all $\alpha \in C_c^\infty(G : F)$ for which $supp(\alpha) \subseteq \{x \in G : \sigma(x) \leq t(s)\}$ we have that

$$||\mathfrak{F}_H(\alpha)(Q : \chi : \nu)|| \leq C_N(1 + |\nu|)^{-N} e^{s|\Re\nu|},$$

where the inequality holds for all $Q \in P(A)$, $\chi \in \hat{M}(F)$ and $\nu \in \mathfrak{a}_{\mathbb{C}}^*$. □

Now set $x = k\exp(X)$, $k \in K$, $X \in \mathfrak{s}$ and consider the twin *elementary spherical functions* $\Xi(x) = \int_K e^{-\rho_Q(H_Q(x))}dk$ and $\sigma(x) = ||X||$ (Trombi [25.],

p. 100). It is clear that $\sigma(xy) \leq \sigma(x) + \sigma(y)$ and $\sigma(x^{-1}) = \sigma(x)$ for all $x, y \in G$ (Varadarajan [29.], p. 167). A collection of continuous seminorms may be placed on $C^\infty(G)$ given as

$$v^p_{a,r}(\alpha) = \sup_{x \in G} |\alpha(x : a)| \Xi^{-2/p}(x)(1 + \sigma(x))^r,$$

for $\alpha \in C^\infty(G)$, $a \in \mathfrak{C}$, $r \in \mathbb{R}$, $0 < p \leq 2$; through which we then define the L^p–Schwartz space on G (and of Harish-Chandra type) as

$$\mathcal{C}^p(G) = \{\alpha \in C^\infty(G) : v^p_{a,r}(\alpha) < \infty, \forall\, a \in \mathfrak{C}, r \in \mathbb{R}\}.$$

It may be shown that $\mathcal{C}^p(G) \subset L^p(G)$ ([25.], p. 100 and [25.], p. 231) and that $C^\infty_c(G) \subset \mathcal{C}^p(G)$, where these inclusions are continuous and with dense images ([29.], p. 341).

An understanding of these spaces depend on the asymptotic estimate of Ξ. The basic result concerning the asymptotic behaviour of Ξ on $G = SL(2, \mathbb{R})$ is found as follow.

1.3 Proposition. (Varadarajan [29.], p. 228) There are constants c_1, c_2 such that (as $t \to \infty$)

$$\Xi(a_t)e^t = c_1 t + c_2 + O(t^4 e^{-3t}).$$

In particular, there is a constant $c > 0$ such that

$$e^{-t} \leq \Xi(a_t) \leq ce^{-t}(1 + t), \quad t \geq 0. \square$$

The above $SL(2, \mathbb{R})$–result is generalized to all G as follows.

1.4 Theorem. (Gangolli and Varadarajan [14.], p. 161) Let d be the number of reflecting hyperplanes in $W(A)$ (equivalently, the number of short positive roots). Then we can find a constant $C > 0$ such that

$$e^{-\rho(\log a)} \leq \Xi(a) \leq Ce^{-\rho(\log a)}(1 + ||\log a||)^d$$

for all $a \in cl(A^+)$. $\square$

With these $\mathcal{C}^p(G : F)$ can be shown to be a convolution subalgebra of $\mathcal{C}^p(G)$, both of which are Fréchet algebras with respect to the collection of (continuous) seminorms above. If $0 < p < q \leq 2$, then we have that

$$C^\infty_c(G) \subset \mathcal{C}^p(G) \subset \mathcal{C}^q(G) \subset \mathcal{C}^2(G)$$

which also restricts to

$$C_c^\infty(G:F) \subset \mathcal{C}^p(G:F) \subset \mathcal{C}^q(G:F) \subset \mathcal{C}^2(G:F).$$

We shall denote the space of cusp forms in $\mathcal{C}^2(G)$ by ${}^o\mathcal{C}^2(G)$ $(= \emptyset$, whenever $rk(G) > rk(K))$, set ${}^o\mathcal{C}^p(G) =^o \mathcal{C}^2(G) \cap \mathcal{C}^p(G)$ and define (its orthogonal subspace as)

$$\mathcal{C}_H^p(G) = \{g \in \mathcal{C}^p(G) : \int_G \alpha(x)\bar{g}(x)dx = 0, \forall \alpha \in^o \mathcal{C}^p(G)\}.$$

In close conformity with the definition of $\mathfrak{F}(\alpha)$ above we shall have that

$$\mathcal{C}^p(G) = \begin{cases} \mathcal{C}_H^p(G), & \text{if } rk(G) > rk(K), \\ {}^o\mathcal{C}^p(G) \oplus \mathcal{C}_H^p(G), & \text{if } rk(G) = rk(K). \end{cases}$$

Asymptotic growth of all the members of $\mathcal{C}^p(G)$ follows from those given in Lemma 1.2 and Theorem 1.4. The major pioneering result of Arthur (as a consequence of these various estimates and asymptotic growth) is therefore the following.

1.5 Theorem. (Arthur [3.], p. 2.3 and 3.2) The map $\mathfrak{F} : f \mapsto \mathfrak{F}(f)$, for $f \in C_c^\infty(G)$, extends uniquely to an isometry from $L^2(G)$ (respectively, from $\mathcal{C}^2(G)$) onto $L^2(\hat{G})$ (respectively, onto $\mathcal{C}^2(\hat{G})$.) If $\mathcal{C}^2(G)'$ denotes the topological dual space of $\mathcal{C}^2(G)$, then the transpose map $\mathcal{C}^2(\hat{G})' \dashrightarrow \mathcal{C}^2(G)'$ is a topological isomorphism. □

The above is the $p = 2$ version of Trombi's theorem. Our aim in this paper is to extend Trombi's results ([25.]) on the operator-valued Fourier transform $\alpha \mapsto \mathfrak{F}(\alpha)$ (proven only for each $\alpha \in \mathcal{C}^p(G:F)$) to include all of $\alpha \in \mathcal{C}^p(G)$ (See Lemma 1, p. 102 and p. 85 of [25.]) and to characterize its Fourier image.

The Fourier image of $\alpha \in \mathcal{C}^p(G)$ or its subalgebra $\alpha \in \mathcal{C}^p(G:F)$, for some values of p and/or some G, had been characterized by several authors notably Ehrenpreis and Mautner ([10.], [11.], [12.]), Langlands ([19.]), Arthur ([1.], [2.], [3.]), Harish-Chandra ([16.]), Trombi-Varadarajan ([28.]), Eguchi ([9.]), Trombi ([25], [26.]), Barker ([5.]), Varadarajan ([29.]) and lately Oyadare ([20.], [21.], [22.]). See a synopsis of these results in the next section. The closest of these results to the complete characterization of the

operator-valued Fourier transform of all $\mathcal{C}^p(G)$, $0 < p \leq 2$, as a Fréchet convolution Schwartz algebra are those of Trombi ([25.]) which he established for the Fréchet convolution Schwartz algebra $\mathcal{C}^p(G : F)$ in which $F \subset \hat{K}$ with $|F| < \infty$. We shall show in this paper that, through a decomposition of $\hat{K}$ into a countable union of finite sets (in §3.) and using a technique of Varadarajan ([29.] successfully employed for $G = SL(2, \mathbb{R})$), the Fréchet convolution Schwartz algebra $\mathcal{C}^p(G)$ can be decomposed into a direct sum of the subalgebras $\mathcal{C}^p(G : F)$ (in §4.), therefore using these decompositions to give the needed complete characterization of the operator-valued Fourier image of $\mathcal{C}^p(G)$, here denoted as $\mathcal{C}^p(\hat{G})$. It is then easy to study the structural details of $\mathcal{C}^p(\hat{G})$ as a Fréchet multiplication algebra.

The significance of this research is inherent in its applicatons to Bandlimited Signals ([7.]), W^p-spaces ([23.]), Quantum circuit on Qudits ([13.]) and to Schwartz spaces of Chebli-Trimeche Hypergroups ([6.]).

§2. Major Results on Fourier Transform on G.

Historically the three papers of Ehrenpreis and Mautner ([10.], [11.], [12.]) marked the beginning of the construction of the Schwartz function space on a semisimple Lie group as well as the consideration of the Fourier transform of such function space. In these papers only the Lie group $G = SL(2, \mathbb{R})$ (in its Lorentz realization) was considered and its harmonic analysis carried out using the point of view of the representation theory and differential equations introduced by Bargmann ([4.]). Thereafter Langlands employed these equations to derive Plancherel related theorems for $SL(2, \mathbb{R})$ in an unpublished manusript of 1962. Taking a cue from here, Harish-Chandra ([19.]) introduced his Schwartz space of a general semisimple Lie group, which formed the basis for Arthur's 1970 thesis ([1.], under the guidance of Langlands) and his two other unpublished manuscripts where a detailed analysis is accorded the Schwartz space and its Fourier transform.

Arthur started out with the consideration of the Harish-Chandra Schwartz space $\mathcal{C}^2(G)$ built (from $C^\infty(G)$) on the L^2-space of G of real-rank 1 in [1.] and of any arbitrary real-rank in [2.] and [3.]. The real-rank 1 case was attained by Arthur by first seeking to derive a Schwartz-space Plancherel formula for G ([1.], p. $30 - 35$) at both $x = 1$ (the identity of G) and at any $x \in G$, which then suggested the form and decomposition format of the Fourier image $\mathcal{C}^2(\hat{G})$, while much of the work on the arbitrary real-rank G was done ([2.] and [3.]) via the parametrization of some parabolic subgroups.

Arthur's major theorem is that: the map

$$\mathfrak{F} : f \mapsto \hat{f}$$

gives a topological isomorphism of $\mathcal{C}^2(G)$ onto $\mathcal{C}^2(\hat{G})$ (followed by its tempered distribution consequences). It thus became clear from Arthur's papers that harmonic analysis of the Schwartz space of any of such groups G must entail the elucidation of elementary (or $\tau-$) spherical functions, (all) irreducible unitary representations and the asymptotic estimates/growth of their matrix coefficients (via the analysis of their perturbed differential equations).

Arthur did not however consider example by example of G; did not impose any further restriction on $\mathcal{C}^2(G)$; did not consider $\mathcal{C}^p(G)$, for $p \neq 2$ and did not view the L^2-Schwartz space as a Fréchet convolution Schwartz algebra. Nevertheless, the singular fact that $\mathcal{C}^p(G) \subseteq \mathcal{C}^2(G)$ for all $0 < p \leq 2$, makes Arthur's thesis and the two other manuscripts crucial to every other result proved for the L^p-Schwartz space $\mathcal{C}^p(G)$ on G. A major property of these spaces is its algebraic nature and, with the collection of seminorms earlier defined above, we could prove that it is a topological algebra in the following manner.

2.1 Theorem. (Gangolli and Varadarajan [14.], p. 255) The space $\mathcal{C}^2(G)$ is an algebra under convolution and $\mathcal{C}^2(G//K)$ is a commutative sub-algebra. The map $(f, g) \mapsto f * g$ from $\mathcal{C}^2(G) \times \mathcal{C}^2(G)$ to $\mathcal{C}^2(G)$ is continuous. □

At about the same time (in the year 1970) that Arthur submitted his Yale thesis, P. C. Trombi also submitted a thesis on almost the same subject, but in his own case Trombi considered the general $p-$case of $0 < p \leq 2$ for the spherical Schwartz spaces $\mathcal{C}^p(G//K)$ as a Fréchet convolution Schwartz algebra for split rank one G. Having realized that the successful theory of the Schwartz space provides the correct framework for all questions of harmonic analysis on G, Trombi took it as a lifetime project to characterize the entire $\mathcal{C}^p(G)$, without any restriction on p or on G or on $\mathcal{C}^p(G)$. A first look at the Harish-Chandra Fourier transform, along this line of generalization, is its correct definition and its control by the seminorms on $\mathcal{C}^p(G)$ given as follows.

2.2 Theorem. (Gangolli and Varadarajan [14.], p. 262) For any measurable function f satisfying the strong inequality, the integral

$$(\mathfrak{F}f)(\lambda) = \int_G f(x)\varphi_\lambda(x)dx$$

converges absolutely and uniformly for all $\lambda \in \mathfrak{F}_I$ and is a continuous function on $\mathfrak{F}_I$. If $r \geq 0$ is such that $c := \int_G \Xi^2(1+\sigma)^{-r}dx < \infty$, then

$$|\mathfrak{F}f(\lambda)| \leq cv_{1,r}(f),$$

for all $\lambda \in \mathfrak{F}_I$. $\square$

This is a compatibility result between the Fourier transform map and members of $\mathcal{C}^p(G)$.

First the restriction to G with split rank one was jointly removed by Trombi and Varadarajan ([28.]) for the spherical Schwartz convolution algebra $\mathcal{C}^p(G//K)$ leading to the celebrated Trombi-Varadarajan theorem, without any further restriction on the real reductive group G or on p. We refer the reader to consult Gangolli and Varadarajan ([14.]) to see the limit of the spectral theory on G used in the proof of the Trombi-Varadarajan theorem, a first version of whose main result is the following.

2.3 Theorem. (Trombi-Varadarajan [28.], p. 273) Let $(dx, d\lambda)$ be an admissible pair of Haar measures on G and $\mathcal{F}_I$. Then the Harish-Chandra Fourier transform $\mathfrak{F}$ is a topological algebra isomorphism of the convolution algebra $\mathcal{C}^p(G//K)$ with the multiplication algebra $\mathcal{S}(\mathfrak{F}_I)^{\mathfrak{w}}$. Moreover, we have, for any $f \in \mathcal{C}^p(G//K)$, we have

$$(\mathfrak{F}f)(\lambda) = \int_G f(x)\varphi_\lambda(x)dx, \quad \lambda \in \mathfrak{F}_I$$

and

$$f(x) = |\mathfrak{w}|^{-1} \int_{\mathfrak{F}_I} (\mathfrak{F}f)(\lambda)\varphi_{-\lambda}(x)c(\lambda)^{-1}c(\lambda)^{-1}d\lambda, \quad x \in G. \square$$

If we define $\mathcal{Z}(\mathfrak{F}^0) = \mathcal{S}(\mathfrak{a}^*)$ and, for each $\epsilon > 0$, let $\mathcal{Z}(\mathfrak{F}^\epsilon)$ be the space of all $\mathbb{C}$-valued functions Φ such that (*i.*) Φ is defined and holomorphic on $int(\mathfrak{F}^\epsilon)$, and (*ii.*) for each holomorphic differential operator D with polynomial coefficients we have $\sup_{int(\mathfrak{F}^\epsilon)} |D\Phi| < \infty$. The space $\mathcal{Z}(\mathfrak{F}^\epsilon)$ is converted to a Fréchet algebra by equipping it with the topology generated by the collection, $\|\cdot\|_{\mathcal{Z}(\mathfrak{F}^\epsilon)}$, of seminorms given by

$$\|\Phi\|_{\mathcal{Z}(\mathfrak{F}^\epsilon)} := \sup_{int(\mathfrak{F}^\epsilon)} |D\Phi|.$$

It is known that $D\Phi$ above extends to a continuous function on all of $\mathfrak{F}^\epsilon$ ([28.], pp. 278 – 279). An appropriate subalgebra of $\mathcal{Z}(\mathfrak{F}^\epsilon)$ for our purpose is the closed subalgebra $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$ consisting of $\mathfrak{w}$-invariant elements of $\mathcal{Z}(\mathfrak{F}^\epsilon)$, $\epsilon \geq 0$. The celebrated *Trombi-Varadarajan theorem* is stated below.

2.4 Theorem. (Trombi-Varadarajan [28.]) Let $0 < p \leq 2$ and set $\epsilon = (\frac{2}{p}) - 1$. Then the spherical Harish-Chandra Fourier transform $f \mapsto \widehat{f}$ is a linear topological algebra isomorphism of $\mathcal{C}^p(G//K)$ onto $\bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)$. □

This theorem was effortlessly lifted to include all the *symmetric-space* Schwartz convolution algebra $\mathcal{C}^p(G/K)$ by Eguchi ([9.]) using the analysis of the spectral theory on G/K. The spectral point of view could however not be pushed further to accommodate all of $\mathcal{C}^p(G)$ since the expected spectrum is no longer pure imaginary. (Gangolli and Varadarajan [14.], p. 355). Trombi then introduced the method of first considering the subalgebra $\mathcal{C}^p(G : F)$ consisting of members of $\mathcal{C}^p(G)$ that have their $K-$types in a finite subset F of $\hat{K}$, for both the real-rank 1 real reductive groups in ([25.]) and for any arbitrary real-rank real reductive group via $K-$finite functions of $\mathcal{C}^p(G)$ in an unpublished manuscript (See Barker [5.]). There appeared however to be no way to adapt all their methods to the consideration of the Fourier transform to the entirety of $\mathcal{C}^p(G)$.

In order to get fresh perspectives on the way forward other authors waded in on the subject by making explicit computations of all necessary results for specific groups, notably $G = SL(2,\mathbb{R})$. It is in this wise should Barker's *AMS Memoirs* [5.] on $\mathcal{C}^p(SL(2,\mathbb{R}))$ be seen, where arguments in Arthur's $\mathcal{C}^2(G)-$case were used to remove the *"finite-set membership of the $K-$types"* in Trombi's greatest and most successful efforts. There again (in the explicit $\mathcal{C}^2(SL(2,\mathbb{R}))-$case) the spectrum of the differential equations that are needed to finally lift Trombi's results failed to be pure-imaginary and seemed irreparable. Another author with another fresh idea for the way forward, as explicitly presented in the $SL(2,\mathbb{R})-$case, was Varadarajan [29.] who surreptitiously weaned the arguments from its over-reliance on *"finite-set membership of the $K-$types."* In both of these successful attempts on the entire $\mathcal{C}^p(SL(2,\mathbb{R}))$ (of Barker [5.]) and $\mathcal{C}^2(SL(2,\mathbb{R}))$ (of Varadarajan [29.]) it was clear that the pathway of first considering type-(m,n) spherical functions in [29.] is the most efficient.

Even though Varadarajan [29.] only considered $\mathcal{C}^2(SL(2,\mathbb{R}))$ (out of all

of $\mathcal{C}^p(SL(2,\mathbb{R}))$, his method of conclusion for the characterization of the operator-valued Fourier transform of the Schwartz convolution algebra is unsuspectingly the closest to completing the works of Trombi in extending the properties of the operator-valued Fourier transform from $\mathcal{C}^p(G:F)$ to all of $\mathcal{C}^p(G)$. Varadarajan started out with the scalar problem of $\mathcal{C}^2_{mn}(SL(2,\mathbb{R}))$ of L^2-functions f on G satisfying

$$f(u_{\theta_1}xu_{\theta_2}) = e^{i(m+n)\theta}f(x)$$

(for all $u_{\theta_1}, u_{\theta_2} \in K = SO(2)$ and $x \in SL(2,\mathbb{R})$) known as the type-(m,n) spherical functions. Noting that

$$e^{i(m+n)\theta}f(x) = e^{im\theta}f(x)e^{in\theta} = \tau_m(u_{\theta_1})f(x)\tau_n(u_{\theta_1}),$$

in which we have used the expression $\tau_n(u_\theta) = e^{in\theta}$, for any $K-$type of $G = SL(2,\mathbb{R})$ (Knapp [18.], p. 627), we have the transformation relation

$$f(u_{\theta_1}xu_{\theta_2}) = \tau_m(u_{\theta_1})f(x)\tau_n(u_{\theta_1}).$$

We observe here that, in the language of the general context of Trombi, the finite subset F of $\hat{K}$ is (in the special case of Varadarajan's $SL(2,\mathbb{R})$) simply $F = \{m,n\}$. This is made clearer if we recall that the map

$$f \mapsto \int_K \int_K \bar{\chi}_\xi(k_1)\alpha(\tau_l(k_1)\cdot x\cdot \tau_r(k_2))\bar{\chi}_\xi(k_2))dk_1dk_2$$

is a continuous map of $C_c^\infty(G)$ onto the space $C^\infty_{c:\xi_l,\xi_r}(G)$, of $\tau = (\tau_l,\tau_r)-$ spherical functions on G, given exhaustively as

$$\{f \in C_c^\infty(G) : f(\tau_l(k_1)\cdot x\cdot \tau_r(k_2)) = \xi_l(k_1)f(x)\xi_r(k_2), \forall k_1,k_2 \in K, x \in G\}$$

(Barker [5.], p. 9). Hence, recalling from §1. above, we have that

$$C^\infty_{c:\xi_l,\xi_r}(G) = C_c^\infty(G:\{\xi_l,\xi_r\}) = C_c^\infty(G:F)$$

in the notation of Trombi. Varadarajan [29.] then considered the parameters of the set $F = \{\xi_l,\xi_r\}$ (written as $\{m,n\}$ for $G = SL(2,\mathbb{R})$) as variables, using them (as matrix-parameters) to compute a matrix realization of every member of the operator algebra $\pi(\mathcal{C}^2(SL(2,\mathbb{R})))$. This is the function-space technique we generalize to the consideration of $\tau-$sperical functions on G.

Such functions may be derived via an admissible representation π in which

$$\pi_{|_K} = \sum_{w \in \hat{K}} n_w w,$$

so that if F_1 and F_2 are any two finite subsets of $\hat{K}$ and if $\tau_l := \sum_{w \in F_1} n_w w$ and $\tau_r := \sum_{w \in F_2} n_w w$ in which E_l and E_r are the respective orthogonal projections, then

$$f(x) = E_l \pi(x) E_r$$

is a $\tau = (\tau_l, \tau_r)-$ spherical function on G and an eigenfunction of $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}}))$ (Knapp [18.], p. 234). More explicitly, $\tau_l(k)u := \pi_{_{F_1}}(k)u$ and $\tau_r(k)u := u\pi_{_{F_2}}(k)$ where $\pi_{_F}(k) := E_{_F}\pi(k)E_{_F}$ with

$$E_{_F} = \sum_{w \in \hat{K}} \pi(\xi_w), \quad \text{in which } \xi_w(k) := dim(w) \cdot \chi_w(k^{-1}),$$

χ_w being the character of w. (Gangolli and Varadarajan [14.], p. 32).

Our aim in the present paper is to establish that, for any real reductive group G, we can realize the Fourier image of $\mathcal{C}^p(G)$ as a Fréchet multiplication algebra of infinite block matrices. A summary of the major results of the present section follows, numbered according to chronology.

Summary of Major Results of the Theory.

A. Scalar-valued Fourier tranform on G

For $p = 2$:

(1) L. Ehrenpreis and F. Mautner (1955; 1957; 1959): Fourier analysis of $\mathcal{C}_{mn}(SL(2, \mathbb{R}))$ and $\mathcal{C}(SL(2, \mathbb{R}))$ as Schwartz spaces.

(2) R. P. Langlands (1962): Fourier analysis of $\mathcal{C}(SL(2, \mathbb{R}))$ as a Schwartz space.

For $0 < p \leq 2$:

(3) P. C. Trombi (1970): Fourier analysis of $\mathcal{C}^p(G//K)$, for only split rank 1 groups G, as a Fréchet convolution Schwartz algebra.

(4) P. C. Trombi and V. S. Varadarajan (1971): Fourier analysis of $\mathcal{C}^p(G//K)$ as a Fréchet convolution Schwartz algebra.

(5) M. Eguchi (1979): Fourier analysis of $\mathcal{C}^p(G/K)$ as a Fréchet convolution Schwartz algebra.

(6) P. C. Trombi (1982): Invariant Fourier analysis of $\mathcal{C}^p(G : F)$ as a topological vector space.

(7) O. O. Oyadare (2019; 2022; 2023): Fourier analysis of $\mathcal{C}^p(G)$ as a Fréchet convolution Schwartz algebra, whose Fourier image is proved to be the scalar subalgebra

$$\{(\widehat{\xi_1})^{-1} \cdot h \cdot (\widehat{\xi_1})^{-1} : h \in \bar{\mathcal{Z}}(\mathfrak{F}^\epsilon)\} \subsetneq \mathcal{C}^p(\hat{G}),$$

proof of its Bochner theorem and analysis of the canonical wave-packets on G.

B. Vector-valued Fourier tranform on G

For $p = 2$:

(1) J. Arthur (1970; 1972): Fourier analysis of $\mathcal{C}(G)$ for real-rank 1 groups G and arbitrary real-rank groups as a Schwartz space.

(2) V. S. Varadarajan (1989): Fourier analysis of $\mathcal{C}(SL(2,\mathbb{R}))$ as a Fréchet convolution Schwartz algebra.

For $0 < p \leq 2$:

(3) P. C. Trombi (1981): Fourier analysis of $\mathcal{C}^p(G : F)$ for real-rank 1 groups and later $\mathcal{C}^p(G, K)$ for arbitrary real-rank groups G as a Fréchet convolution Schwartz algebra.

(4) W. H. Barker (1988): Fourier analysis of $\mathcal{C}^p(SL(2,\mathbb{R}))$ and $\mathcal{C}^o(SL(2,\mathbb{R}))$ as Fréchet convolution Schwartz algebras.

(5) O. O. Oyadare (2024): The present paper.

§3. Some Elementary Results on $K-$types.

The major lessons in the 1955 to 1969 works of the masters as reviewed in the earlier sections are that, from Ehrenpreis and Mautner's type-(m, n) to Trombi's finite membership of $K-$types, the $\tau-$spherical functions (in which $\tau = (\tau_l, \tau_r)$ as before) are the gateways to accessing the portals of function space theory (Gangolli and Varadarajan [14.], p. *vii*), that $\tau-$spherical functions are intricately looped with the concept of $K-$types (Knapp [18.], p. 215), that we need not pair up the $K-$types (as done in type$-(m, n)$) and that the entire situation is controlled by the $K-$types of these functions all of which reside in some finite subset of $\hat{K}$. It may therefore be crucial to closely understand the $K-$types of (spherical) functions in order to move forward from Trombi.

There are however a few distinguished members of $\hat{K}$ that have been the subject of research papers and that have been used to understand the different types of representations of G from algebraic point of view. For these we refer to the relevant chapters of Knapp [18.] as well as the references therein for orientation. Our aim in this section (and as would be applied to prove our main results in the next section) is to exploit the detailed structure of $\hat{K}$ in relation to the understanding of $\alpha \in C_c^\infty(G)$ satisfying

$$d(\xi)^2 \int_K \int_K \bar{\chi}_\xi(k_1)\alpha(\tau_l(k_1)\cdot x \cdot \tau_r(k_2))\bar{\chi}_\xi(k_2))dk_1dk_2 = \alpha(x),$$

for all $k_1, k_2 \in K$, $x \in G$ and $\xi \in F \subset \hat{K}$ with $|F| < \infty$.

Let us recall that $\hat{K}$ denote the set of equivalence classes of (necessarily finite-dimensional) irreducible unitary representations of K. The following first result on $\hat{K}$ is a bit difficult to notice in the literature.

3.1 Lemma. $\hat{K}$ is a countable set. □

We refer to Warner [32.], p. 250 for the proof of Lemma 3.1 and note that a careful look at the detailed verification of Peter-Weyl theorem suggests a knowledge of Lemma 3.1. See also Wallach [31.], p. 36 and Dixmier [8.], §15. The content of this Lemma is that the set $\hat{K}$ is equivalent to any one (or to a finite or a countable union) of countable sets like $\mathbb{N}, \cdots, \mathbb{Z}, \cdots \mathbb{Q}, \cdots$ of the field of number theory.

3.2 Examples. (1) For $G = SL(2,\mathbb{R})$ we have $K = SO(2)$. Here $\hat{K} \simeq \mathbb{Z}$ where $n \in \mathbb{Z}$ corresponds to the representation

$$\tau_n : u_\theta \mapsto \tau_n(u_\theta) = e^{in\theta}$$

(Knapp [18.], p. 627).

(2) For $G = SL(2,\mathbb{C})$ we have $K = SU(2)$ and $\hat{K} = \{n \in \mathbb{Z} : n \geq 0\}$.

Lemma 3.1 also allows us to use the elementary properties of a countable set when employing the set $\hat{K}$. One of such properties is the elementary fact that every countable set is a countable (respectively, finite) union of some finite (respectively, countable) subsets. In the special case of $\hat{K}$ we have the following.

3.3 Corollary. Let K and $\hat{K}$ be as above. Then there exists $F_\alpha \subseteq \hat{K}$, where α is in some countable indexing set Ω and $|F| < \infty$, such that

$$\hat{K} = \bigcup_{\alpha \in \Omega} F_\alpha.$$

Proof. We already have from §1 that

$$\hat{K} = \bigcup_{\xi_{j^*}} F(\xi_{j^*}),$$

where ξ_{j^*} is in some subset of $\hat{K}$. Lemma 3.1 simply assures that this subset of $\hat{K}$ is indeed countable. □

The finite set F_α may be taken as a singleton or as $F_\alpha = \{m_\alpha, n_\alpha\}$, etc. We have already seen the case of $F_\alpha = \{m_\alpha, n_\alpha\}$ in the type-(m_α, n_α) point of view as it was unconsciously undertaken in Ehrenpreis and Mautner ([10.]), Barker ([5.]) and Varadarajan ([29.]). We may specifically refer to the projections

$$\mathcal{C}^2(G) \to \mathcal{C}^2_{mn}(G)$$

in the proof of Theorem 57 of [29.] and of Proposition 5.1 in [5.]. The direct import of these elementary results on $\mathcal{C}^p(G)$ is our major interest.

3.4 Proposition. Every $K-$type of members of $\mathcal{C}^p(G)$ is in a finite subset of $\hat{K}$.

Proof. Since the projections

$$\mathcal{C}^p(G) \to \mathcal{C}^p(G : F)$$

is continuous via the map

$$\alpha(x) \mapsto d(\xi)^2 \int_K \int_K \bar{\chi}_\xi(k_1)\alpha(\tau_l(k_1) \cdot x \cdot \tau_r(k_2))\bar{\chi}_\xi(k_2))dk_1 dk_2,$$

the result follows from Corollary 3.3. □

§4. $K-$type Decomposition of $\mathcal{C}^p(G)$ and $\mathcal{C}^p(\hat{G})$ for real-rank 1.

The motivating result of this paper as proved in the last section is the following.

4.1 Lemma. For any G of arbitrary real-rank,

$$\mathcal{C}^p_{\xi_l,\xi_r}(G) = \mathcal{C}^p(G:F)_{|_{F=\{\xi_l,\xi_r\}}}. \square$$

Now let G be or real-rank 1. We need the space $\mathcal{C}^p(G:F)$, for $F \subset \hat{K}$ with $|F| < \infty$. It has been shown that $\mathcal{C}^p(G:F)$ is a Fréchet convolution Schwartz algebra which is explicitly seen as

$$\mathcal{C}^p(G:F) = \begin{cases} \mathcal{C}^p_H(G:F), & \text{if } rk(G) > rk(K), \\ \mathcal{C}^p_B(G:F) \oplus \mathcal{C}^p_H(G:F), & \text{if } rk(G) = rk(K). \end{cases}$$

We refer to §1 and §2 for some unexplained notations and to Trombi [25.], p. 108 for the meanings of the spaces $\mathcal{C}^p_B(\hat{G}:F)$ and $\mathcal{C}^p_H(\hat{G}:F)$.

The main result of Trombi [25.] is that the operator-valued Fourier transform $\mathfrak{F}$ (as defined in §1) is a linear topological algebra isomorphism of $\mathcal{C}^p(G:F)$ (under convolution) onto $\mathcal{C}^p(\hat{G}:F)$, where $\mathcal{C}^p(\hat{G}:F)$ is the Schwartz space of matrix-functions (under multiplication) in which

$$\mathcal{C}^p(\hat{G}:F) = \begin{cases} \mathcal{C}^p_H(\hat{G}:F), & \text{if } rk(G) > rk(K), \\ \mathcal{C}^p_B(\hat{G}:F) \otimes \mathcal{C}^p_H(\hat{G}:F), & \text{if } rk(G) = rk(K), \end{cases}$$

which are linear subspaces of functions $g = (g_B, g_H)$ for which

$$g_B(\Lambda, \Lambda, M) = \sum_{i' \in I'_P} \mathcal{F}(\alpha_{i'})(\Lambda, \Lambda, M) g_H(i')$$

for all $\alpha_{i'} \in \mathcal{C}^p(G:F)$ and $\Lambda \in L^+_B$ such that

$$|\Lambda(H_\beta)| \leq (\frac{2}{p} - 1)k(\beta)$$

(for some positive non-compact root β of $(\mathfrak{g}, \mathfrak{b})$ with $k(\beta) := \frac{1}{2}\sum_{\alpha \in P} |\alpha(\bar{H}_\beta)|$ where P is a positive root system for $\triangle(\mathfrak{g}, \mathfrak{b})$ for which $\beta(\bar{H}_\beta) = 2$). It is noted here that $g_B(\Lambda, \Lambda, M)$ denotes the matrix representation of the operator $g_B(\Lambda)$ relative to the basis

$$\{\phi_{\gamma,l}(\Lambda) : \gamma \in F, 1 \leq l \leq n(\Lambda:\gamma)\}$$

while the matrix $g_H(i')$ is relative to the basis

$$\{\phi_{\gamma,m} : \gamma \in F, 1 \leq l \leq n(\chi : \gamma)\}.$$

The induced topology from the tensor product topology on

$$\mathcal{C}_B^p(\hat{G} : F) \otimes \mathcal{C}_H^p(\hat{G} : F)$$

converts $\mathcal{C}^p(\hat{G} : F)$ into a Schwartz Fréchet space and, by virtue of the Fourier map,

$$\mathfrak{F} : \mathcal{C}^p(G : F) \to \mathcal{C}^p(\hat{G} : F),$$

we conclude that $\mathcal{C}^p(\hat{G} : F)$ is therefore a Schwartz Fréchet multiplication algebra.

4.2 Proposition. Let $F \subset \hat{K}$ with $|F| < \infty$. Then

$$\bigoplus_{F \subset \hat{K}} \mathcal{C}^p(G : F)$$

is dense in $\mathcal{C}^p(G)$.

Proof. We employ Theorem 4.4.2.1 (on p. 261) and Theorem 4.4.2.5 (on p. 265) of Warner [32.] and the Remark of Warner [32.], p. 161 to conclude that for every $\alpha \in \mathcal{C}^p(G)$, the sum

$$\sum_{F \subset \hat{K}, |F| < \infty} \bar{\chi}_F * \alpha * \bar{\chi}_F$$

converges absolutely to α. Hence the proof. □

4.3 Remarks.

(1) Proposition 4.2 above provides a technical way of handling any problem or gap that could be created by the existence of functions $\alpha \in \mathcal{C}^p(G)$ with a $K-$type not in a finite subset of $\hat{K}$, if such functions ever exist (See Proposition 3.4). It simply says that any of such functions (if they exist) could be taken care of by a countable sum of members of $\mathcal{C}^p(G : F)$. For $p = 2$ and $G = SL(2, \mathbb{R})$, no such function exists and we have (in this example) that

$$\bigoplus_{F \subset \hat{K}, |F| < \infty} \mathcal{C}^2(G : F) = \bigoplus_{m,n \in \mathbb{Z}} \mathcal{C}^2_{mn}(G) = \mathcal{C}^2_{mn}(G),$$

Varadarajan [29.], p. $280 - 281$. Indeed, a readily available finite set F for a clear understanding of the proof of Proposition 4.2 is $F = \{\xi_1, \xi_2\}$ for any $\xi_1, \xi_1 \in \hat{K}$.

(2) The proof of Proposition 4.2 shows that every function $\alpha \in \mathcal{C}^p(G)$ has a unique expansion into the sum

$$\alpha = \sum_{F \subset \hat{K}, |F| < \infty} \bar{\chi}_F * \alpha * \bar{\chi}_F$$

in $\bigoplus_{F \subset \hat{K}, |F| < \infty} \mathcal{C}^p(G : F)$, where we may denote the convolution $\bar{\chi}_F * \alpha * \bar{\chi}_F$ simply by α_F. Now as each $\mathcal{C}^p(G : F)$ splits into a direct sum given explicitly as

$$\mathcal{C}^p_B(G : F) \oplus \mathcal{C}^p_H(G : F)$$

(if $rk(G) = rk(K)$) or as $\mathcal{C}^p_H(G : F)$ (if $rk(G) > rk(K)$), it follows that α_F is a sum which may be given as $\alpha_F = \alpha_{B,F} + \alpha_{H,F}$, where $\alpha_{B,F} \in \mathcal{C}^p_B(G : F)$ and $\alpha_{H,F} \in \mathcal{C}^p_H(G : F)$. Hence, each $\alpha \in \mathcal{C}^p(G)$ may be uniquely written as $\alpha = \alpha_B + \alpha_H$, with

$$\alpha_B = \sum_{F \subset \hat{K}, |F| < \infty} \bar{\chi}_F * \alpha_B * \bar{\chi}_F$$

and

$$\alpha_H = \sum_{F \subset \hat{K}, |F| < \infty} \bar{\chi}_F * \alpha_H * \bar{\chi}_F.$$

4.4 Corollary. Let $F \subset \hat{K}$ with $|F| < \infty$. Then

$$\bigoplus_{F \subset \hat{K}, |F| < \infty} \mathcal{C}^p(G : F)$$

is a dense Fréchet convolution Schwartz subalgebra of $\mathcal{C}^p(G)$. □

We may now consider the (tensor) product of Trombi's images of the Fourier transform of each of $\mathcal{C}^p(G : F)$, as suggested by the $SL(2, \mathbb{R})-$versions of Barker [5.] and Varadarajan [29.].

4.5 Proposition. Let $F \subset \hat{K}$ with $|F| < \infty$. Then

$$\bigotimes_{F \subset \hat{K}, |F| < \infty} \mathcal{C}^p(\hat{G} : F)$$

is a dense Fréchet multiplication Schwartz subalgebra of $\mathcal{C}^p(\hat{G})$.

Proof. Each of

$$\mathfrak{F}^F : \mathcal{C}^p(G : F) \to \mathcal{C}^p(\hat{G} : F)$$

is a linear topological algebra isomorphism, Trombi [25.], p. 122. We now define the map $\mathfrak{F}$ on $\bigoplus_{F \subset \hat{K}, |F| < \infty} \mathcal{C}^p(G : F)$ as

$$\mathfrak{F}(\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots) := \mathfrak{F}^{F_1}(\check{\alpha}_1) \otimes \mathfrak{F}^{F_2}(\check{\alpha}_2) \otimes \cdots \otimes \mathfrak{F}^{F_n}(\check{\alpha}_n) \otimes \cdots,$$

where $F_j \subset \hat{K}$, $|F_j| < \infty$ and $\alpha_j \in \mathcal{C}^p(G : F_j)$. See Barker [5.], p. 44 for the $SL(2, \mathbb{R})$ version of $\mathfrak{F}$. The result follows from Corollary 4.4, since each of

$$\mathfrak{F}^F : \mathcal{C}^p(G : F) \to \mathcal{C}^p(\hat{G} : F)$$

is a linear topological algebra isomorphism. □

We are now in a position to state the fundamental theorem of harmonic analysis on any G of real-rank 1 via the map $\mathfrak{F}$ in the proof of Proposition 4.5.

4.6 Theorem. The Fourier transform

$$\mathfrak{F} : \mathcal{C}^p(G) \to \mathcal{C}^p(\hat{G}) : \alpha \mapsto \mathfrak{F}(\check{\alpha})$$

is a linear topological algebra isomorphism of the Schwartz Fréchet algebras.

Proof. The result is an immediate consequence of Corollary 4.4 and Proposition 4.5. □

4.7 Remarks.

(1) The canonical wave-packet corresponding to every $\alpha \in \mathcal{C}^p(G)$ in the proof of Theorem 4.6 is here denoted by $\phi_{\mathfrak{F}(\alpha)}$ and is the function on G given as

$$\phi_{\mathfrak{F}(\alpha)}(x) = \sum_{\Lambda \in L_B^+(F)} d(\Lambda) \phi_{\mathfrak{F}_B(\alpha)}(\Lambda : x) + \sum_{\chi \in \hat{M}(F)} d(\chi) \phi_{\mathfrak{F}_H(\alpha)}(Q : \chi : x),$$

where

$$\phi_{\mathfrak{F}_B(\alpha)}(\Lambda : x) = \sum_{F\subset\hat{K},|F|<\infty} tr\{\mathfrak{F}_B^F(\alpha)(\Lambda)\pi_\Lambda^F(x)\}$$

and

$$\phi_{\mathfrak{F}_H(\alpha)}(Q : \chi : x) = \sum_{F\subset\hat{K},|F|<\infty} tr\{\mathfrak{F}_H^F(\alpha)(Q : \chi : \nu)\pi_{Q:\chi:\nu}^F(x)\}\mu^F(\chi : \nu)$$

are the corresponding wave-packets in $\mathcal{C}_B^p(G : F)$ and $\mathcal{C}_H^p(G : F)$, respectively. (See Trombi [25.], p. 85). See also Barker [5.], p. $49-50$.

(2) The construction of the dense Fréchet convolution Schwartz subalgebra in Corollary 4.4 is in line with the splitting of $\mathcal{C}^p(G)$ in Barker [5.], p. 44 for $G = SL(2,\mathbb{R})$. It thus follows that the Fourier transform $\mathfrak{F}$ maps a direct sum in $\mathcal{C}^p(G)$ (on)to a tensor product in $\mathcal{C}^p(\hat{G})$. This suggests a construction of a basis in both $\mathcal{C}^p(G)$ and $\mathcal{C}^p(\hat{G})$.

A consideration of the basis

$$\phi_{\gamma,m} \otimes \phi_{\gamma,l},$$

with $\gamma \in F$ and $(l,m) \in \mathbb{Z}^2$ of the space

$$\mathcal{C}^p(\hat{G} : F) = \begin{cases} \mathcal{C}_H^p(\hat{G} : F), & \text{if } rk(G) > rk(K), \\ \mathcal{C}_B^p(\hat{G} : F) \otimes \mathcal{C}_H^p(\hat{G} : F), & \text{if } rk(G) = rk(K), \end{cases}$$

reveals that the members of $\mathcal{C}^p(\hat{G} : F)$ could be realized as matrices of the form

$$(\mathfrak{F}_B(\check{\alpha})_{(\gamma,m)}(\Lambda) \otimes \mathfrak{F}_H(\check{\alpha})_{(\gamma,l)}(Q : \chi : \nu))_{\gamma,M},$$

where $\gamma \in F$ and $M := (l,m) \in \mathbb{Z}^2$. We can therefore explicitly realize $\mathcal{C}^p(\hat{G})$ as an algebra of infinite matrices with a countable number of blocks as follows.

4.8 Corollary. The Fourier image $\mathcal{C}^p(\hat{G})$ (of $\mathcal{C}^p(G)$ under the map $\mathfrak{F}$) is a Schwartz Fréchet multiplication algebra consisting of block matrices of the form

$$((\mathfrak{F}_B(\check{\alpha})_{(\gamma,m)}(\Lambda) \otimes \mathfrak{F}_H(\check{\alpha})_{(\gamma,l)}(Q : \chi : \nu))_{\gamma\in F,(l,m)\in\mathbb{Z}^2})_{F\subset\hat{K},|F|<\infty}.$$

Proof. The matrix realization follows from combining Corollary 3.3, Proposition 4.5, the above remark on a basis of $\mathcal{C}^p(\hat{G} : F)$ and Theorem

4.6. □

The reader may consult Varadarajan [29.], p. 276 – 278 for the Hilbert-space version of the last corollary when $p = 2$ and $G = SL(2,\mathbb{R})$. In this special (linear) case, the block-matrix of Corollary 4.8 reduces (or is isomorphic) to the partitioned block-matrix

$$((\int f\bar{\phi}_{kmn}dG),(\int f\bar{f}_{mn}(\mu:\cdot)dG))_{m,n\in\mathbb{Z}}$$

for $f \in \mathcal{C}^2(SL(2,\mathbb{R}))$ as contained in Varadarajan [29.].

Now if $\mathcal{C}^p(G)'$ and $\mathcal{C}^p(\hat{G})'$ denote the topological dual spaces of $\mathcal{C}^p(G)$ and $\mathcal{C}^p(\hat{G})$, respectively, with the weak topology, the transpose $\mathfrak{F}'$ (See the end-matters of Arthur [1]) of $\mathfrak{F}$ in Theorem 4.4 exists and is equally a linear topological isomorphism

$$\mathcal{C}^p(\hat{G})' \to \mathcal{C}^p(G)'.$$

This isomorphism affords the development of harmonic analysis for $p-$tempered distributions on groups G of real-rank 1. See also Oyadare [20.] and [21].

The success of the methods used above hinges on the earlier established fact that almost all members of the space $\mathcal{C}^p(G)$ may be realized as a countable sum of some $\tau = (\tau_l, \tau_r)-$spherical functions on G. This allowed us to use some density properties of the space of these functions to arrive at our conclusion for real-rank 1 groups G. These same density properties are actually available for any G of any arbitrary rank and we shall explore the existence of such density results in arriving at a theorem similar to Theorem 4.6 for such G via the space of $K-$finite functions in $\mathcal{C}^p(G)$ in the next section.

§5. **Arbitrary real-rank case for** G.

Now let G denote any real reductive group of arbitrary real-rank. We shall refer to Varadarajan [30.] for the structure theory of G. The method employed for the arbitrary real-rank case is to combine the techniques of Varadarajan [29.] and Gårding [15.]. We recall that for any representation π of G on a vector space V, the first known proof (if not the only one still known) that verifies that the subspace $C^\infty(\pi)$ (of V) consisting of smooth vectors for π is dense in V, is indirectly through the use of what is now known as the Gårding subspace $\mathcal{G}(\pi)$. The indirect proof (of the denseness of $C^\infty(\pi)$

in V) was achieved in two simple steps:

$$\mathcal{G}(\pi) \subseteq C^{\infty}(\pi) \subseteq V \quad \text{and} \quad \mathcal{G}(\pi) \text{ is dense in } V.$$

The argument being put forward in this paper is analogous to the method of L. Gårding [15.]. Indeed, in its technical details, it is the same as using the approach of type$-(m,n)$ spherical functions as done by Varadarajan [29.] to create subspaces whose direct sum in $\mathcal{C}^p(G)$ (respectively, tensor product in $\mathcal{C}^p(\hat{G})$) is dense in $\mathcal{C}^p(G)$ (respectively, in $\mathcal{C}^p(\hat{G})$). The first of these series of results is the following.

5.1 Proposition. We have that

$$\mathcal{C}^p_{\tau_l,\tau_r}(G) \subseteq \mathcal{C}^p(G:F) \subseteq \mathcal{C}^p(G),$$

for any τ_l, τ_r in some F in $\hat{K}$.

Proof. It is clear that $\{\tau_l, \tau_r\}$ is a finite subset of $\hat{K}$ on which the Trombi analysis of [25.] restricts to. □

5.2 Theorem. The direct sum $\bigoplus_{\tau_l,\tau_r \in \hat{K}} \mathcal{C}^p_{\tau_l,\tau_r}(G)$ is dense in $\mathcal{C}^p(G)$.

Proof. Every $f \in \mathcal{C}^p(G)$ has a representation as

$$f = \sum_{F \subset \hat{K}, |F| < \infty} \bar{\chi}_F * f * \bar{\chi}_F,$$

where each pair $\{\tau_l, \tau_r\} \subset F$. □

5.3 Theorem. The direct sum $\bigoplus_{F \subseteq \hat{K}, |F| < \infty} \mathcal{C}^p(G:F)$ is dense in $\mathcal{C}^p(G)$.

Proof. This follows from Theorem 5.2. □

Now denote by $\mathcal{C}^p_{\tau_l,\tau_r}(\hat{G})$, $\mathcal{C}^p(\hat{G}:F)$ and $\mathcal{C}^p(\hat{G})$ the respective images of $\mathcal{C}^p_{\tau_l,\tau_r}(G)$, $\mathcal{C}^p(G:F)$ and $\mathcal{C}^p(G)$ under the operator-valued Fourier transform $\alpha \mapsto \mathfrak{F}(\check{\alpha})$. The fact that $\mathfrak{F}$ is a linear topological isomorphism given as

$$\mathcal{C}^p_{\tau_l,\tau_r}(G) \to \mathcal{C}^p_{\tau_l,\tau_r}(\hat{G})$$

and

$$\mathcal{C}^p(G:F) \to \mathcal{C}^p(\hat{G}:F)$$

in Trombi [25.] imply the following result.

5.4 Theorem. We always have that

$$\mathcal{C}^p_{\tau_l,\tau_r}(\hat{G}) \subseteq \mathcal{C}^p(\hat{G} : F) \subseteq \mathcal{C}^p(\hat{G}),$$

for any τ_l, τ_r in some F and that $\bigotimes_{\tau_l,\tau_r \in \hat{K}} \mathcal{C}^p_{\tau_l,\tau_r}(\hat{G})$ is dense in $\mathcal{C}^p(\hat{G})$. Hence $\bigotimes_{F \subset \hat{K}, |F| < \infty} \mathcal{C}^p(\hat{G} : F)$ is dense in $\mathcal{C}^p(\hat{G})$. □

We then have the full version of Theorem 4.6 given below.

5.5 Theorem. The Fourier transform

$$\mathfrak{F} : \mathcal{C}^p(G) \to \mathcal{C}^p(\hat{G}) : \alpha \mapsto \mathfrak{F}(\breve{\alpha})$$

is a linear topological algebra isomorphism of the Schwartz Fréchet algebras. □

This leads to an explicit expression for members of $\mathcal{C}^p(\hat{G})$ in terms of the earlier basis given in §4. This is explicitly given below.

5.6 Corollary. The Fourier image $\mathcal{C}^p(\hat{G})$ (of $\mathcal{C}^p(G)$ under the map $\mathfrak{F}$) is a Schwartz Fréchet multiplication algebra consisting of block matrices of the form

$$\left((\mathfrak{F}_B(\breve{\alpha})_{(\gamma,m)}(\Lambda) \otimes \mathfrak{F}_H(\breve{\alpha})_{(\gamma,l)}(Q : \chi : \nu))_{\gamma \in F, (l,m) \in \mathbb{Z}^2} \right)_{F \subset \hat{K}, |F| < \infty}. \square$$

§6. Coda

Another approach to the general theory is the following, based on Trombi's $K-$finite results. Consider the Fréchet convolution Schwartz algebra $\mathcal{C}^p(G,K)$ consisting of $K-$finite members of $\mathcal{C}^p(G)$. It has been shown by Trombi (See Barker [5.]) that

$$\mathfrak{F} : \mathcal{C}^p(G,K) \to \mathcal{C}^p(\hat{G},K)$$

is a linear topological algebra isomorphism of Fréchet algebras. Let us now define a subalgebra of $\mathcal{C}^p(G)$ denoted as $\mathcal{C}^p(G,K:F)$ explicitly as

$$\mathcal{C}^p(G,K:F) := \mathcal{C}^p(G,K) \bigcap \mathcal{C}^p(G:F).$$

This is the subalgebra of all $K-$finite members of $\mathcal{C}^p(G)$ whose $K-$types lie in a finite subset F of $\hat{K}$. $\mathcal{C}^p(G,K:F)$ is topologized by the induced continuous

seminorms of $\mathcal{C}^p(G,K)$. We equally define $\mathcal{C}^p_H(G,K:F)$, $\mathcal{C}^p_H(G,K:F)$ and $\mathcal{C}^p(G,K:F)$ as in the earlier case of real-rank 1. We have the decomposition of $\mathcal{C}^p(G,K:F)$ (respectively, $\mathcal{C}^p(\hat{G},K:F) := \mathfrak{F}(\mathcal{C}^p(G,K:F))$) into the direct sum (respectively, tensor product) of the space $\mathcal{C}^p_H(G,K:F)$ with $\mathcal{C}^p_B(G,K:F)$ or with 0 (respectively, $\mathcal{C}^p_H(\hat{G},K:F)$ with $\mathcal{C}^p_B(\hat{G},K:F)$ or with 0), depending on whether $rk(G) = rk(K)$ or $rk(G) > rk(K)$, respectively. It is clear that

$$\bigoplus_{F\subset\hat{K},|F|<\infty} \mathcal{C}^p(G,K:F) = \mathcal{C}^p(G,K),$$

which by Trombi (in Barker [5.]) is dense in $\mathcal{C}^p(G)$. We then have a more general version of Proposition 4.5 in that for $F \subset \hat{K}$ with $|F| < \infty$,

$$\bigotimes_{F\subset\hat{K},|F|<\infty} \mathcal{C}^p(\hat{G},K:F)$$

is the Fréchet multiplication algebra $\mathcal{C}^p(\hat{G},K)$ and is dense in $\mathcal{C}^p(\hat{G})$. This also leads to the fundamental theorem on G as follows.

6.1 Theorem. The Fourier transform

$$\mathfrak{F} : \mathcal{C}^p(G) \to \mathcal{C}^p(\hat{G}) : \alpha \mapsto \mathfrak{F}(\breve{\alpha})$$

is a linear topological algebra isomorphism of the Fréchet convolution Schwartz algebras.

Proof. Each $\mathfrak{F} : \mathcal{C}^p(G,K:F) \to \mathcal{C}^p(\hat{G},K:F)$ is a linear topological algebra isomorphism. Since $\bigoplus_{F\subset\hat{K},|F|<\infty} \mathcal{C}^p(G,K:F) = \mathcal{C}^p(G,K)$ is dense in $\mathcal{C}^p(G)$ and $\bigotimes_{F\subset\hat{K},|F|<\infty} \mathcal{C}^p(\hat{G},K:F) = \mathcal{C}^p(\hat{G},K)$ is dense in $\mathcal{C}^p(\hat{G})$, then the result. follows. □

The proof of the Bochner theorem and analysis of its canonical wave-packets on G, corresponding to the generalization contained in both Theorems 4.6 (for real-rank 1 groups G) and Theorem (5.5 or) 6.1 (for arbitrary real-rank groups G) is essentially as established in Oyadare [21.], [22.] with necessary basis-adjustment (from the scalar version to the vector version) as required.

Declarations.

Ethics Approval and consent to participate: I declare that there is no unethical conduct in the research and I also give my consent to participate in Ethics Approval.

Consent for publication: I give my consent to the publication of this manuscript.

Availability of data and materials: Materials consulted during the research are as listed in the References below.

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Authors' contributions: I declare that I was the only one who contributed to the conduct of this research.

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A NOTE ON PSEUDOSYMMETRIC KENMOTSU MANIFOLDS SATISFYING CERTAIN CONDITIONS

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Abstract

The purpose of this paper is to study pseudosymmetric and Ricci pseudosymmetric of a Kenmotsu manifolds. We obtain the necessary and sufficient conditions for a Kenmotsu manifold, W_2 pseudosymmetric, W_2 Ricci pseudosymmetric, W_3 pseudosymmetric and W_3 Ricci pseudosymmetric. Finally, some interesting results on Kenmotsu manifolds are obtained.

AMS Classification 53C15, 53C25, 53C35
Keywords. Kenmotsu manifold, W_2-curvature tensor, W_3 curvature tensor

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1. Introduction

In a Riemannian manifold, the Riemannian curvature tensor is R and for each $\beta_1, \beta_2 \in \chi(M)$, if $R(\beta_1, \beta_2) \cdot R = 0$, then the manifold is said to be semisymmetric. Similarly, if $R(\beta_1, \beta_2) \cdot S = 0$, the manifold is called Ricci semisymmetric, if $R(\beta_1, \beta_2) \cdot P = 0$, the manifold is called projective semisymmetric, $R(\beta_1, \beta_2) \cdot \widetilde{Z} = 0$ and the manifold is called concircular semisymmetric, where S is the Ricci curvature tensor, P is the projective curvature tensor and $\widetilde{Z}$ is the concircular curvature tensor. Studies on the symmetric Riemannian manifolds started with Cartan [5]. In the following periods, many authors have studied the curvature tensors of various manifolds [4, 6, 10, 14, 15].

In [13] is studied on a classification of almost $C(a)-$manifolds. T. Mert researched pseudosymmetric and Ricci pseudosymmetric almost $C(a)-$manifolds. For an almost $C(a)-$manifold, Riemannian pseudosymmetric, Riemannian Ricci pseudosymmetric, Ricci pseudosymmetric, projective pseudossymmetric, projective Ricci pseudosymmetric, concircular pseudosymmetric and concircular Ricci pseudosymmetric cases are considered and new results are obtained.

K. Kenmotsu researched a class of a contact Riemannian manifold and call them Kenmotsu manifold [12]. He studied that if Kenmotsu manifold satisfies the condition $R(\beta_1, \beta_2).R = 0$, then the manifold is of negative curvature -1, where R is the Riemannian curvature tensor of type $(1,3)$ and $R(\beta_1, \beta_2)$ denotes the derivation of the tensor algebra at each point of the tangent space. The properties of Kenmotsu manifolds have been studied by several authors[7, 8, 19, 21, 22].

In this article, we have researched the pseudosymmetric and Ricci pseudosymmetric of Kenmotsu manifold. For Kenmotsu manifold, W_2 pseudosymmetric, W_2 Ricci pseudosymmetric, W_3 pseudosymmetric and W_3 Ricci pseudosymmetric cases are considered. Then some results are obtained and classifications have been made.

2. Preliminaries

Let M be a $(2n+1)-$dimensional almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is an $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and the Riemanniann metric g on M satisfy

$$\phi^2(\beta_1) = -\beta_1 + \eta(\beta_1)\xi, \quad \eta(\phi\beta_1) = 0, \tag{2.1}$$

$$\eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi) = 0 \tag{2.2}$$

for all $\beta_1, \beta_2 \in \chi(M)$ [16]. Let g be Riemannian metric compatible with (ϕ, ξ, η), that is

$$g(\phi\beta_1, \phi\beta_2) = g(\beta_1, \beta_2) - \eta(\beta_1)\eta(\beta_2), \tag{2.3}$$

or equivalently,

$$g(\beta_1, \phi\beta_2) = -g(\phi\beta_1, \beta_2) \quad \text{and} \quad g(\beta_1, \xi) = \eta(\beta_1) \tag{2.4}$$

for all $\beta_1, \beta_2 \in \chi(M)$ [2]. If in addition to above relation

$$(\nabla_{\beta_1}\phi)\beta_2 = -\eta(\beta_2)\phi\beta_1 - g(\beta_1, \phi\beta_2)\xi, \tag{2.5}$$

and

$$\nabla_{\beta_1}\xi = \beta_1 - \eta(\beta_1)\xi, \tag{2.6}$$

where ∇ denotes the Riemannian connection of g hold, then $M(\phi, \xi, \eta, g)$ is called Kenmotsu manifold. Kenmotsu manifold becomes a Kenmotsu manifold if

$$g(\beta_1, \phi\beta_2) = d\eta(\beta_1, \beta_2). \tag{2.7}$$

In a Kenmotsu manifold M, the following relation holds [12, 8]:

$$(\nabla_{\beta_1}\eta)\beta_2 = g(\beta_1, \beta_2) - \eta(\beta_1)\eta(\beta_2), \tag{2.8}$$

$$R(\beta_1, \beta_2)\xi = \eta(\beta_1)\beta_2 - \eta(\beta_2)\beta_1, \tag{2.9}$$

$$R(\xi, \beta_1)\beta_2 = \eta(\beta_2)\beta_1 - g(\beta_1, \beta_2)\xi, \tag{2.10}$$

$$S(\beta_1, \xi) = -2n\eta(\beta_1), \tag{2.11}$$

$$Q\xi = -2n\xi, \tag{2.12}$$

where R is the Riemannian curvature tensor and S is Ricci tensor defined by $S(\beta_1, \beta_2) = g(Q\beta_1, \beta_2)$, where Q is Ricci operator. It yields to

$$S(\phi\beta_1, \phi\beta_2) = S(\beta_1, \beta_2) + 2n\eta(\beta_1)\eta(\beta_2). \tag{2.13}$$

Definition 2.1. A Kenmotsu manifold M is said to be an $\eta-$Einstein manifold if its Ricci tensor S of the form

$$S(\beta_1, \beta_2) = \alpha g(\beta_1, \beta_2) + \beta\eta(\beta_1)\eta(\beta_2) \tag{2.14}$$

for arbitrary vector fields β_1, β_2; where α and β are functions on (M^{2n+1}, g). If $\beta = 0$, then $\eta-$ Einstein manifold becomes Einstein manifold [3].

On a semi-Riemannian manifold (M,g), for a $(0,k)-$type tensor field $(0,k)$-type tensor field T and $(0,2)$-type tensor field A, $(0,k+2)$-type Tachibana tensor field $Q(A,T)$ is defined as

$$\begin{aligned} Q(A,T)(\beta_{11},\beta_{12},...,\beta_{1k};\beta_1,\beta_2) &= -T((\beta_1 \wedge_A \beta_2)\beta_{11},\beta_{12},...,\beta_{1k}) \\ &- T(\beta_{11},(\beta_1 \wedge_A \beta_2)\beta_1,\beta_{13},...,\beta_{1k}) \\ &\cdot \\ &\cdot \\ &\cdot \\ &- T(\beta_{11},\beta_{12},...,(\beta_1 \wedge_A \beta_2)\beta_{1k}), \end{aligned} \tag{2.15}$$

for all $\beta_{11},\beta_{12},...,\beta_{1k},\beta_1,\beta_2 \in \chi(M)$, where

$$(\beta_1 \wedge_A \beta_2)\beta_3 = A(\beta_2,\beta_3)\beta_1 - A(\beta_1,\beta_3)\beta_2. \tag{2.16}$$

for all $\beta_1,\beta_2,\beta_3 \in \chi(M)$.

3. Characterization of Pseudosymmetric Kenmotsu manifold

In this section, the case of pseudosymmetry and Ricci pseudosymmetry of Kenmotsu manifold are investigated. According to W_2 curvature tensor and W_3 curvature tensor, the pseudosymmetrical and Ricci pseudosymmetrical cases of the Kenmotsu manifold can be given as follows.

Let (M,g) be an $(2n+1)$-dimensional Riemannian manifold. Then the W_2 curvature tensor is defined by [18]. Furthermore, W_2 the curvature tensor for Riemannian manifold (M^{2n+1},g) is given by

$$W_2(\beta_1,\beta_2)\beta_3 = R(\beta_1,\beta_2)\beta_3 - \frac{1}{2n}[g(\beta_2,\beta_3)Q\beta_1 - g(\beta_1,\beta_3)Q\beta_2] \tag{3.1}$$

for all $\beta_1,\beta_2,\beta_3 \in \chi(M)$ [18]. If we choose, respectively, $\beta_1 = \xi$ and $\beta_3 = \xi$ in (3.1), then we obtain as follows:

$$W_2(\xi,\beta_2)\beta_3 = \eta(\beta_3)\beta_2 + \frac{1}{2n}\eta(\beta_3)Q\beta_2, \tag{3.2}$$

$$W_2(\beta_1,\beta_2)\xi = \eta(\beta_1)\beta_2 - \eta(\beta_2)\beta_1 - \frac{1}{2n}(\eta(\beta_2)Q\beta_1 - \eta(\beta_1)Q\beta_2). \tag{3.3}$$

In addition, we choose $\beta_3 = \xi$ in (3.2), we obtain as follows:

$$W_2(\xi,\beta_2)\xi = \beta_2 + \frac{1}{2n}Q\beta_2. \tag{3.4}$$

Definition 3.1. Let M be Kenmotsu manifold with $(2n+1)-$dimensional, R be the Riemannian curvature tensor of M, S be the Ricci curvature tensor of M and W_2 be the W_2 curvature tensor.

(i) If the pair $R \cdot W_2$ and $Q(g, W_2)$ are linearly dependent, that is, if a λ_1 function can be found on the set
$M_1 = \{\beta_1 \in M | g(\beta_1) \neq W_2(\beta_1)\}$ such that

$$R.W_2 = \lambda_1 Q(g, W_2), \tag{3.5}$$

the M manifold is called a W_2 pseudosymmetric manifold.

(ii) If the pair $R \cdot W_2$ and $Q(S, W_2)$ are linearly dependent, that is, if a λ_2 function can be found on the set
$M_2 = \{\beta_1 \in M | S(\beta_1) \neq W_2(\beta_1)\}$ such that

$$R.W_2 = \lambda_2 Q(S, W_2), \tag{3.6}$$

the M manifold is called a W_2 Ricci pseudosymmetric manifold. Particularly, if $\lambda_1 = 0$, then this manifold is said to be semisymmetric [9].

Let us now investigate the cases of W_2 pseudosymmetry and W_2 Ricci pseudosymmetry.

Theorem 3.2. *If a $(2n+1)-$dimensional M Kenmotsu manifold is a W_2 pseudosymmetric manifold, then M is an Einstein manifold.*

Proof. Let us suppose that Kenmotsu manifold M is a W_2 pseudosymmetric manifold. Then, we can write

$$(R(\beta_1, \beta_2) \cdot W_2)(\beta_3, \beta_4, \beta_5) = \lambda_1 Q(g, W_2)(\beta_5, \beta_4, \beta_3; \beta_1, \beta_2), \tag{3.7}$$

for each $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in \chi(M)$. This means that

$$\begin{aligned} & R(\beta_1, \beta_2)W_2(\beta_3, \beta_4)\beta_5 - W_2(R(\beta_1, \beta_2)\beta_3, \beta_4)\beta_5 - W_2(\beta_3, R(\beta_1, \beta_2)\beta_4)\beta_5 \\ & -W_2(\beta_3, \beta_4)R(\beta_1, \beta_2)\beta_5 \\ = \; & -\lambda_1\{W_2((\beta_1 \wedge_g \beta_2)\beta_5, \beta_4)\beta_3 + W_2(\beta_5, (\beta_1 \wedge_g \beta_2)\beta_4)\beta_3 \\ & +W_2(\beta_5, \beta_4)(\beta_1 \wedge_g \beta_2)\beta_3\}. \end{aligned} \tag{3.8}$$

Here taking $\beta_1 = \beta_3 = \xi$ and by using (2.9), (2.10) and (2.16) in (3.8), we reach at

$$\begin{aligned} & \eta(W_2(\xi, \beta_4)\beta_5)\beta_2 - g(\beta_2, W_2(\xi, \beta_4)\beta_5) - R(\beta_2 - \eta(\beta_2)\xi, \beta_4)\beta_5 \\ & -W_2(\xi, \eta(\beta_4)\beta_2 - g(\beta_4, \beta_2)\xi)\beta_5 - \eta(W_2(\xi, \beta_2)\beta_5)\beta_4 + g(\beta_4, R(\xi, \beta_2)\beta_5) \\ = \; & -\lambda_1\{W_2(g(\beta_2, \beta_5)\xi - \eta(\beta_5)\beta_2, \beta_4)\xi + W_2(\beta_5, g(\beta_2, \beta_4)\xi - \eta(\beta_4)\beta_2) \\ & +W_2(\beta_5, \beta_4)(\eta(\beta_2)\xi - \beta_2)\}. \end{aligned} \tag{3.9}$$

If we use (3.2), (3.3) and taking $\beta_5 = \xi$ in (3.9) and make the necessary abbreviations, then we have

$$\begin{aligned}
& \eta(\beta_4 + \frac{Q\beta_4}{2n})\beta_2 - g(\beta_2, \beta_4 + \frac{Q\beta_4}{2n})\xi - W_2(\beta_2, \beta_4)\xi + \eta(\beta_2)W_2(\xi, \beta_4)\xi \\
& -\eta(\beta_4)W_2(\xi, \beta_2)\xi - W_2(\xi, \beta_4)\beta_2 + \eta(\beta_2)W_2(\xi, \beta_4)\xi \\
= \; & -\lambda_1\{\eta(\beta_2)(\beta_4 + \frac{Q\beta_4}{2n}) - (\eta(\beta_2)\beta_4 - \eta(\beta_4)\beta_2 - \eta(\beta_4)\frac{Q\beta_2}{2n} \\
& +\eta(\beta_2)\frac{Q\beta_4}{2n}) - \eta(\beta_4)(\beta_2 + \frac{Q\beta_2}{2n}) + \eta(\beta_2)(\beta_4 + \frac{Q\beta_4}{2n}) \\
& -W_2(\xi, \beta_4)\beta_2\}. \qquad (3.10)
\end{aligned}$$

Taking the inner product with $\xi \in \chi(M)$ on both sides of (3.10) and make use of (3.2), then we can infer

$$S(\beta_4, \beta_2) = -2ng(\beta_4, \beta_2).$$

This completes our proof. □

Theorem 3.3. *If a $(2n+1)-$dimensional M Kenmotsu manifold is a W_2 Ricci pseudosymmetric manifold, then M is an Einstein manifold.*

Proof. Let us assume that Kenmotsu manifold M is a W_2 Ricci pseudosymmetric manifold. This implies that

$$(R(\beta_1, \beta_2) \cdot W_2)(\beta_3, \beta_4, \beta_5) = \lambda_2 Q(S, W_2)(\beta_5, \beta_4, \beta_3; \beta_1, \beta_2), \qquad (3.11)$$

for each $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in \chi(M)$, that is,

$$\begin{aligned}
& R(\beta_1, \beta_2)W_2(\beta_3, \beta_4)\beta_5 - W_2(R(\beta_1, \beta_2)\beta_3, \beta_4)\beta_5 - W_2(\beta_3, R(\beta_1, \beta_2)\beta_4)\beta_5 \\
& -W_2(\beta_3, \beta_4)R(\beta_1, \beta_2)\beta_5 \\
= \; & -\lambda_2\{W_2((\beta_1 \wedge_S \beta_2)\beta_5, \beta_4)\beta_3 + W_2(\beta_5, (\beta_1 \wedge_S \beta_2)\beta_4)\beta_3 \\
& +W_2(\beta_5, \beta_4)(\beta_1 \wedge_S \beta_2)\beta_3\}. \qquad (3.12)
\end{aligned}$$

Here, taking $\beta_1 = \beta_3 = \xi$ and using (2.9), (2.16) in (3.12), we arrive at

$$\begin{aligned}
& \eta(W_2(\xi, \beta_4)\beta_5)\beta_2 - g(\beta_2, W_2(\xi, \beta_4)\beta_5) - W_2(\beta_2 - \eta(\beta_2)\xi, \beta_4)\beta_5 \\
& -W_2(\xi, \eta(\beta_4)\beta_2 - g(\beta_4, \beta_2)\xi)\beta_5 - \eta(W_2(\xi, \beta_2)\beta_5)\beta_4 + g(\beta_4, W_2(\xi, \beta_2)\beta_5) \\
= \; & -\lambda_2\{W_2(S(\beta_2, \beta_5)\xi + 2n\eta(\beta_5)\beta_2, \beta_4)\xi + W_2(\beta_5, S(\beta_2, \beta_4)\xi \\
& +2n\eta(\beta_4)\beta_2)\xi + 2nW_2(\beta_5, \beta_4)(\beta_2 - \eta(\beta_2)\xi)\}. \qquad (3.13)
\end{aligned}$$

If ξ is taken of β_5 at (3.13), considering (2.11), (3.2), then we get

$$\begin{aligned}
&\eta(\beta_4+\frac{Q\beta_4}{2n})\beta_2-g(\beta_2,\beta_4+\frac{Q\beta_4}{2n})\xi-W_2(\beta_2,\beta_4)\xi+\eta(\beta_2)W_2(\xi,\beta_4)\xi\\
&-\eta(\beta_4)W_2(\xi,\beta_2)\xi-W_2(\xi,\beta_4)\beta_2+\eta(\beta_2)W_2(\xi,\beta_4)\xi\\
=\;&-\lambda_2\{-4n\eta(\beta_2)(\beta_4+\frac{Q\beta_4}{2n})+2n(\eta(\beta_2)\beta_4-\eta(\beta_4)\beta_2-\eta(\beta_4)\frac{Q\beta_2}{2n}\\
&+\eta(\beta_2)\frac{Q\beta_4}{2n})+2n\eta(\beta_4)(\beta_2+\frac{Q\beta_2}{2n})+2nW_2(\xi,\beta_4)\beta_2\}. \qquad (3.14)
\end{aligned}$$

Inner product both sides of (3.14) by $\xi\in\chi(M)$ and make the necessary adjustments, we obtain

$$S(\beta_4,\beta_2)=-2ng(\beta_4,\beta_2). \qquad (3.15)$$

This proves our assertion. □

Let (M,g) be an $(2n+1)$-dimensional Riemannian manifold. Then the W_3 curvature tensor is defined by [18]. Furthermore, W_3 the curvature tensor for Riemannian manifold (M^{2n+1},g) is given by

$$W_3(\beta_1,\beta_2)\beta_3=R(\beta_1,\beta_2)\beta_3-\frac{1}{2n}[S(\beta_1,\beta_3)\beta_2-g(\beta_2,\beta_3)Q\beta_1] \qquad (3.16)$$

for all $\beta_1,\beta_2,\beta_3\in\chi(M)$ [18]. If we choose, respectively, $\beta_1=\xi$ and $\beta_3=\xi$ in (3.16), then we obtain as follows:

$$W_3(\xi,\beta_2)\beta_3=2\eta(\beta_3)\beta_2-g(\beta_2,\beta_3)\xi, \qquad (3.17)$$

$$W_3(\beta_1,\beta_2)\xi=2\eta(\beta_1)\beta_2-\eta(\beta_2)\beta_1+\eta(\beta_2)\frac{Q\beta_1}{2n}. \qquad (3.18)$$

In addition, we choose $\beta_3=\xi$ in (3.17), we obtain as follows:

$$W_3(\xi,\beta_2)\xi=2\beta_2-2\eta(\beta_2)\xi. \qquad (3.19)$$

Definition 3.4. Let M be Kenmotsu manifold with $(2n+1)-$dimensional, R be the Riemannian curvature tensor of M, S be the Ricci curvature tensor of M and W_3 be the W_3-curvature tensor.

(i) If the pair $R\cdot W_3$ and $Q(g,W_3)$ are linearly dependent, that is, if a λ_3 function can be found on the set

$M_3=\{\beta_1\in M|g(\beta_1)\neq W_3(\beta_1)\}$ such that

$$R.W_3=\lambda_3Q(g,W_3), \qquad (3.20)$$

the M manifold is called a W_3 pseudosymmetric manifold.

(ii) If the pair $R\cdot W_3$ and $Q(S,W_3)$ are linearly dependent, that is, if a λ_4 function can be found on the set

$M_4=\{\beta_1\in M|S(\beta_1)\neq W_3(\beta_1)\}$ such that

$$R.W_3=\lambda_4Q(S,W_3), \qquad (3.21)$$

the M manifold is called a W_3 Ricci pseudosymmetric manifold [9].

Let us now investigate the cases of W_3 pseudosymmetry and W_3 Ricci pseudosymmetry.

Theorem 3.5. *If a $(2n+1)-$dimensional M Kenmotsu manifold is a W_3 pseudosymmetric manifold, then M is a semisymmetric manifold.*

Proof. Let us suppose that Kenmotsu manifold M is a W_3 pseudosymmetric manifold. Then, we can write

$$(R(\beta_1,\beta_2)\cdot W_3)(\beta_3,\beta_4,\beta_5)=\lambda_3 Q(g,W_3)(\beta_5,\beta_4,\beta_3;\beta_1,\beta_2), \tag{3.22}$$

for each $\beta_1,\beta_2,\beta_3,\beta_4,\beta_5\in\chi(M)$. This means that

$$\begin{aligned}
&R(\beta_1,\beta_2)W_3(\beta_3,\beta_4)\beta_5-W_3(R(\beta_1,\beta_2)\beta_3,\beta_4)\beta_5-W_3(\beta_3,R(\beta_1,\beta_2)\beta_4)\beta_5\\
&-W_3(\beta_3,\beta_4)R(\beta_1,\beta_2)\beta_5\\
=\;&-\lambda_3\{W_3((\beta_1\wedge_g\beta_2)\beta_5,\beta_4)\beta_3+W_3(\beta_5,(\beta_1\wedge_g\beta_2)\beta_4)\beta_3\\
&+W_3(\beta_5,\beta_4)(\beta_1\wedge_g\beta_2)\beta_3\},
\end{aligned} \tag{3.23}$$

that is, in the last equality taking $\beta_1=\beta_3=\xi$ and using (2.9), (2.10) and (3.17) in (3.23), we obtain

$$\begin{aligned}
&\eta(W_3(\xi,\beta_4)\beta_5)\beta_2-g(\beta_2,W_3(\xi,\beta_4)\beta_5)-W_3(\beta_2-\eta(\beta_2)\xi,\beta_4)\beta_5\\
&-W_3(\xi,\eta(\beta_4)\beta_2-g(\beta_4,\beta_2)\xi)\beta_5-\eta(W_3(\xi,\beta_2)\beta_5)\beta_4+g(\beta_4,W_3(\xi,\beta_2)\beta_5)\\
=\;&-\lambda_3\{W_3(g(\beta_2,\beta_5)\xi-\eta(\beta_5)\beta_2,\beta_4)\xi+W_3(\beta_5,g(\beta_2,\beta_4)\xi-\eta(\beta_4)\beta_2)\\
&+W_3(\beta_5,\beta_4)(\eta(\beta_2)\xi-\beta_2)\}.
\end{aligned} \tag{3.24}$$

If we use (3.17), (3.18) and taking $\beta_5=\xi$ in (3.24) and make the necessary abbreviations, then we have

$$\begin{aligned}
&\eta(2\beta_4-2\eta(\beta_4)\xi)\beta_2-g(\beta_2,2\beta_4-2\eta(\beta_4)\xi)\xi-W_3(\beta_2,\beta_4)\xi\\
&+\eta(\beta_2)W_3(\xi,\beta_4)\xi-\eta(\beta_4)W_3(\xi,\beta_2)\xi-W_3(\xi,\beta_4)\beta_2\\
&+\eta(\beta_2)W_3(\xi,\beta_4)\xi\\
=\;&-\lambda_3\{\eta(\beta_2)(2\beta_4-2\eta(\beta_4)\xi)-(2\eta(\beta_2)\beta_4-\eta(\beta_4)\beta_2\\
&+\eta(\beta_4)\frac{Q\beta_2}{2n})-\eta(\beta_4)(2\beta_2-2\eta(\beta_2)\xi)+\eta(\beta_2)(2\beta_4\\
&-2\eta(\beta_4)\xi)-(2\eta(\beta_2)\beta_4-2g(\beta_4,\beta_2)\xi)\}.
\end{aligned} \tag{3.25}$$

Taking the inner product with $\xi\in\chi(M)$ on both sides of (3.25) and make use of (3.19), then we arrive

$$2\lambda_3\left[g(\beta_4,\beta_2)-\eta(\beta_4)\eta(\beta_2)\right]=0. \tag{3.26}$$

On the other hand, we know that from (3.26) and (2.3), we conclude

$$2\lambda_3 g(\phi\beta_4,\phi\beta_2)=0.$$

This completes our proof. □

Theorem 3.6. *If a $(2n+1)-$dimensional M Kenmotsu manifold is a W_3 Ricci pseudosymmetric manifold, then M is a semisymmetric manifold.*

Proof. Let us assume that Kenmotsu manifold M is a W_3 Ricci pseudosymmetric manifold. This implies that

$$(R(\beta_1,\beta_2)\cdot W_3)(\beta_3,\beta_4,\beta_5) = \lambda_4 Q(S,W_3)(\beta_5,\beta_4,\beta_3;\beta_1,\beta_2), \tag{3.27}$$

for each $\beta_1,\beta_2,\beta_3,\beta_4,\beta_5 \in \chi(M)$, that is,

$$\begin{aligned} & R(\beta_1,\beta_2)W_3(\beta_3,\beta_4)\beta_5 - W_3(R(\beta_1,\beta_2)\beta_3,\beta_4)\beta_5 - W_3(\beta_3,R(\beta_1,\beta_2)\beta_4)\beta_5 \\ & -W_3(\beta_3,\beta_4)R(\beta_1,\beta_2)\beta_5 \\ = \; & -\lambda_4\{W_3((\beta_1\wedge_S\beta_2)\beta_5,\beta_4)\beta_3 + W_3(\beta_5,(\beta_1\wedge_S\beta_2)\beta_4)\beta_3 \\ & +W_3(\beta_5,\beta_4)(\beta_1\wedge_S\beta_2)\beta_3\}. \end{aligned} \tag{3.28}$$

Taking $\beta_1=\beta_3=\xi$ and using (2.9), (3.17) in (3.28), we have

$$\begin{aligned} & \eta(W_3(\xi,\beta_4)\beta_5)\beta_2 - g(\beta_2,W_3(\xi,\beta_4)\beta_5) - W_3(\beta_2-\eta(\beta_2)\xi,\beta_4)\beta_5 \\ & -W_3(\xi,\eta(\beta_4)\beta_2 - g(\beta_4,\beta_2)\xi)\beta_5 - \eta(W_3(\xi,\beta_2)\beta_5)\beta_4 + g(\beta_4,W_3(\xi,\beta_2)\beta_5) \\ = \; & -\lambda_4\{W_3(S(\beta_2,\beta_5)\xi + 2n\eta(\beta_5)\beta_2,\beta_4)\xi + W_3(\beta_5,S(\beta_2,\beta_4)\xi \\ & +2n\eta(\beta_4)\beta_2)\xi + 2nW_3(\beta_5,\beta_4)(\beta_2-\eta(\beta_2)\xi)\}. \end{aligned} \tag{3.29}$$

If we use (3.17), (3.19) and setting $\beta_5=\xi$ in (3.29), then we get

$$\begin{aligned} & \eta(2\beta_4-2\eta(\beta_4)\xi)\beta_2 - g(\beta_2,2\beta_4-2\eta(\beta_4)\xi)\xi - W_3(\beta_2,\beta_4)\xi \\ & +\eta(\beta_2)W_3(\xi,\beta_4)\xi - \eta(\beta_4)W_3(\xi,\beta_2)\xi - W_3(\xi,\beta_4)\beta_2 \\ & +\eta(\beta_2)W_3(\xi,\beta_4)\xi \\ = \; & -\lambda_4\{-4n\eta(\beta_2)(2\beta_4-2\eta(\beta_4)\xi) + 2n(2\eta(\beta_2)\beta_4 - \eta(\beta_4)\beta_2 \\ & +\eta(\beta_4)\frac{Q\beta_2}{2n}) + 2n(2\eta(\beta_2)\beta_4 - 2g(\beta_4,\beta_2)\xi) \\ & +2n\eta(\beta_4)(2\beta_2-2\eta(\beta_2)\xi)\}. \end{aligned} \tag{3.30}$$

Inner product both sides of (3.30) by $\xi\in\chi(M)$, we obtain

$$4n\lambda_4\left[g(\beta_4,\beta_2)-\eta(\beta_4)\eta(\beta_2)\right]=0.$$

Consequently, it follows from (2.3)

$$4n\lambda_4 g(\phi\beta_4,\phi\beta_2)=0,$$

which proves our assertion. □

4. CONCLUSION

The aim of this paper is to study pseudosymmetric and Ricci pseudosymmetric of a Kenmotsu manifolds. We obtain the necessary and sufficient conditions for a Kenmotsu manifold, W_2 pseudosymmetric, W_2 Ricci pseudosymmetric, W_3 pseudosymmetric and W_3 Ricci pseudosymmetric.

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A CLASS OF POWER ASSOCIATIVE LCC-LOOPS AND SOME ASSOCIATED TOTAL INNER MAPPING GROUP QUESTIONS*†

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Abstract

In this paper, we investigated the algebraic properties of a class of LCC-loops (and its dual) which is power associative, tagged 'twist' of LWPC (RWPC) viz. LTWC (RTWC). Its relationship with Frute and crazy loop was discovered. LTWC-loop was shown to be a class of loop which falls into the Syrbu and Grecu (2019) first answer to a question of Stanovský and Vojtěchovský (2014)-minimal generating set for the total inner mapping group of a loop. In addition, we showed that the class of LTWC-loop also provides an alternative solution to 2014 question and in fact, the inner automorphism group of a LTWC-loop is generated by just one inner mapping. Generalized forms of the left (right) alternative property and flexibility property were introduced and studied in this class of loop. We also explore the properties of the left and right translation of this class of loop and proved some results useful for the construction a finite loop belonging to this same class.

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1 Introduction

The LWPC and the RWPC loops were introduced by Phillips in [16]. George et al. [5] showed that each of these two loop identities described by

$$\begin{aligned} (xy \cdot x) \cdot xz &= x((yx \cdot x)z) \qquad \text{(LWPC)} \\ zx \cdot (x \cdot yx) &= (z(x \cdot xy))x \qquad \text{(RWPC)} \end{aligned}$$

is power associative. They established that the following equations of identities are true in loops

$$\text{(LWPC)} = \text{(LCC)} + (\mathrm{P}_\lambda) \text{ and } \text{(RWPC)} = \text{(RCC)} + (\mathrm{P}_\rho), \text{ where}$$

$$\underbrace{(xy \cdot x)x = x(yx \cdot x)}_{P_\lambda} \qquad \underbrace{x(x \cdot yx) = (x \cdot xy)x}_{P_\rho}$$

George and Jaíyéọlá [6] investigated two identities (Q_{12} and Q_7) similar to LWPC and RWPC. These identities may be seen as 'twist' of LWPC and RWPC, a possible name for them might be LTWC and RTWC.

$$\begin{aligned} x((x \cdot yx)z) &= (x \cdot xy) \cdot xz \qquad \text{(LTWC)} \\ (y(xz \cdot x))x &= yx \cdot (zx \cdot x) \qquad \text{(RTWC)} \end{aligned}$$

Similarly, they showed that the following equations of identities are true in loops

$$\text{(LTWC)} = \text{(LCC)} + (\mathrm{P}_\rho) \text{ and } \text{(RTWC)} = \text{(RCC)} + (\mathrm{P}_\lambda).$$

Likewise, since the equation of identities (LWPC)+(RWPC) = (CC) + (WIP) is true in loops, so also is (LTWC)+(RTWC) = (CC) + (WIP).
In [5], a method of the construction of LWPC-loop (with a necessary and sufficient condition) was developed (see Theorem 3.1 for details) with the aid of a loop (G, R, f) originally constructed in [3, 4] where R is a subgroup of an abelian group G and $f : G \times G \to R$ is a mapping such that $\mathrm{Rad}(f) \geq R$. The semidirect product of the LWPC and RWPC loops were considered in [7].
Below are few generalized forms of identities of Bol-Moufang type of loops

which will be of interest in this current work. Note that some of them have equivalent forms of identities. See Cote et al. [19].

Frute :	$x(x \cdot yz) = yx \cdot xz$	$A1C3$
Frute :	$x(xy \cdot z) = y(x \cdot xz)$	$A2C1$
Crazy :	$(x(xy))z = (yx)(xz)$	$A4C3$
Crazy :	$x(xy \cdot z) = (yx \cdot x)z$	$A2C5$

Frute loops have been studied in Jaiyéọlá et al. [12, 14]. They are Moufang loops while a crazy loop implies any of the Bol-Moufang type loops including groups. A loop has been shown to be a Cheban loop if and only if it is both a left and right Cheban loop in [17].

The purpose of this work is to introduce and study generalized forms of the left (right) alternative property and flexibility properties in LTWC-loop and to show that the inner automorphism group of a LTWC-loop is generated by just one inner mapping.

2 Preliminaries

A loop Q is a set with a binary operation $\cdot$ such that for all $a, b \in Q$, the equations $ax = b$ and $ya = b$ have unique solutions x and y in Q and there exists $e \in Q$ such that $ex = xe = x$ for any $x \in Q$. The unique solutions are given by $x = a \setminus b$ and $y = b/a$. The reader can consult the books [2, 10, 15, 20, 24] for a general background in loop theory.

In a loop Q with identity e, for all $x \in Q$ there exists a unique right inverse element x^ρ and a unique left inverse element x^λ such that $xx^\rho = x^\lambda x = e$. Note that x^ρ is not always equal to x^λ in a loop, when they are equal we write $x^\rho = x^\lambda = x^{-1}$.

In a groupoid $(Q, \cdot)$, for any $x \in Q$, define the right translation (multiplication) map $R_x = R(x)$ and left translation (multiplication) map $L_x = L(x)$ of x by $yR(x) = y \cdot x = yx$ and $yL(x) = x \cdot y = xy$, respectively. Then, $(Q, \cdot)$ is called a quasigroup if the left and right translation maps are bijections. Since the translation maps are bijections in a quasigroup, then the inverse maps $R^{-1}(x)$ and $L^{-1}(x)$ exist and are thus defined by

$yR^{-1}(x) = y/x = xM_y^{-1}$ and $yL^{-1}(x) = x\backslash y = xM_y$. A quasigroup with an identity element is a loop.

A loop $(Q, \cdot)$ is power associative if subloops generated by every single elements are groups. This is easily seen to be equivalent to $x^{m+n} = x^m \cdot x^n$ for every $x \in Q$ and for all $m, n \in \mathbb{Z}$. A loop $(Q, \cdot)$ is said to be 3-power associative if $xx \cdot x = x \cdot xx$ for all $x \in Q$.

Definition 2.1. *Let $(G, \cdot)$ be a loop. G is said to be a left alternative property loop (LAPL) if for all $x, y \in G$, $x \cdot xy = xx \cdot y$, a right alternative property loop (RAPL)if for all $x, y \in G$, $yx \cdot x = y \cdot xx$, and an alternative loop if it is both left and right alternative.*

A loop $(G, \cdot)$ is called a flexible or an elastic loop if the flexibility or elasticity property $xy \cdot x = x \cdot yx$ holds for all $x, y \in G$.

$(G, \cdot)$ is said to have the left inverse property (LIP) if for all $x, y \in G$, $x^\lambda \cdot xy = y$, the right inverse property (RIP) if for all $x, y \in G$, $yx \cdot x^\rho = y$ and the inverse property if it has both left and right inverse properties.

A loop $(G, \cdot)$ is called a weak inverse property (WIP) loop if it satisfies any one of the equivalent identities $x(yx)^\rho = y^\rho$ and $(xy)^\lambda x = y^\lambda$.

Let $n \in \mathbb{Z}$ be fixed. A power associative loop $(G, \cdot)$ is said to have the left n-power alternative (LPA^n) property if for all $x, y \in G$, $\underbrace{x(\cdots(x(x}_{n-\text{times}}y))\cdots) = x^n y$. If a power associative loop $(G, \cdot)$ has the LPA^n-property for all $n \in \mathbb{Z}$, then it is called a left power alternative loop.

Let $n \in \mathbb{Z}$ be fixed. A power associative loop $(G, \cdot)$ is said to have the right n-power alternative (RPA^n) property if for all $x, y \in G$, $(\cdots((y\underbrace{x)x)\cdots)x}_{n-\text{times}} = yx^n$. If a power associative loop $(G, \cdot)$ has the RPA^n-property for all $n \in \mathbb{Z}$, then it is called a right power alternative loop.

A power associative loop is said to be power alternative loop if it is both left and right power alternative loop. Conjugacy closed loops (CC-loop) are loops satisfying the following two identities:

$$zy \cdot x = zx \cdot x\backslash(yx). \qquad (RCC)$$

$$x \cdot yz = (xy)/x \cdot xz. \qquad (LCC)$$

The group of all permutations on a quasigroup G is called the permutation group of G and denoted by $SYM(G)$. The group $\mathcal{M}(G,\cdot) = \langle\{R_x, L_x, \ : \ x \in G\}\rangle$ is called the multiplication group of $(G,\cdot)$ and $\mathcal{M}(G,\cdot) \leq SYM(G)$. The group of automorphisms in $(G,\cdot)$ is denoted by $AUM(G,\cdot)$.

If $e\alpha = e$ in a loop G such that $\alpha \in \mathcal{M}(G)$, then α is called an inner mapping and they form a group $\mathrm{Inn}(G)$ called the inner mapping group. The right, left and middle inner mappings

$$R_{(x,y)} = R_x R_y R_{xy}^{-1} \ , L_{(x,y)} = L_x L_y L_{yx}^{-1} \text{ and } T_x = R_x L_x^{-1}$$

respectively generate the left inner mapping group $\mathrm{Inn}_\lambda(G)$, right inner mapping group $\mathrm{Inn}_\rho(G)$ and the middle inner mapping group $\mathrm{Inn}_\mu(G)$. If

$$\mathrm{Inn}_\lambda(G) \leq AUM(G), \ \mathrm{Inn}_\rho(G) \leq AUM(G), \text{ and } \mathrm{Inn}_\mu(G) \leq AUM(G),$$

then G is called a left A-loop(A_λ-loop), right A-loop(A_ρ-loop) and middle A-loop(A_μ-loop) respectively. It is well known that $\mathrm{Inn}(G,\cdot) = \langle R_{(x,y)}, L_{(x,y)}, T_x \mid x, y \in G\rangle$, but this was later improved to be $\mathrm{Inn}(G,\cdot) = \langle L_{(x,y)}, T_x \mid x, y \in G\rangle$ $= \langle R_{(x,y)}, T_x \mid x, y \in G\rangle$ in Vojtěchovský [23]. If $\mathrm{Inn}(G,\cdot) \leq AUM(G,\cdot)$, then $(G,\cdot)$ is called an automorphic loop (A-loop). Some works in the direction of structure of A-loops can be found in [11, 23].

The group $\mathcal{TM}(G,\cdot) = \langle\{R_x, L_x, M_x \ : \ x \in G\}\rangle$ is called the total multiplication group of $(G,\cdot)$ and an element $\alpha \in \mathcal{TM}(G,\cdot)$ such that $e\alpha = e$ is called a total inner mapping of $(G,\cdot)$ and they form a group $\mathrm{TInn}(G,\cdot)$ called the total inner mapping group. It has been established in Stanovský and Vojtěchovský [21] that $\mathrm{TInn}(G,\cdot) = \langle R_{(x,y)}, L_{(x,y)}, T_x, M_{(x,y)}, U_x \mid x, y \in G\rangle$, where $M_{(x,y)} = M_y M_x M_{y\backslash x}^{-1}$ and $U_x = M_x R_x^{-1}$. Also, Syrbu and Grecu [22] showed that $\mathrm{TInn}(G,\cdot) = \langle R_{(x,y)}, L_{(x,y)}, T_x, P_{(x,y)}, V_x \mid x, y \in G\rangle$

where $P_{(x,y)} = M_x M_y L_x R_y^{-1}$, and $V_x = R_x M_x$. If $\mathrm{TInn}(G,\cdot) \leq AUM(G,\cdot)$, then $(G,\cdot)$ is called a totally automorphic loop (TA-loop). Some other works in the direction of structure of TA-loops can be found in Jaiyéọlá and Effiong [13], Kinyon [18]. As mentioned in [13], [18] raised a question (Question 1.1) of what other interesting varieties of loops can be characterized by specifying that some group of total inner mappings acts as automorphisms? The authors in [13] provided a partial answer to Question 1.1.

Furthermore, the authors in [21] raised a question (Question 1.2 in [13]) if the generating sets of $\mathrm{TInn}(G)$ for a loop G are minimal, i.e, can any

of the five types of mappings be removed? The authors in [13] provided a ostensible answer to Question 1.2. Syrbu and Grecu [22] were able to provide answers to Question 1.2 by showing that:

1. If Q is a power associative loop, then
   $\mathrm{TInn}(Q) = \langle R_{(x,y)}, L_{(x,y)}, P_{(x,y)}, V_x \mid x, y \in Q \rangle$.
2. If Q is a middle Bol loop, then $\mathrm{TInn}(Q) = \langle R_{(x,y)}, P_{(x,y)}, V_x \mid x, y \in Q \rangle$.

In this work, we shall show that LTWC-loop is a class of loop which falls into the Syrbu and Grecu's first answer to Question 1.2. In addition, the class of LTWC-loop also provides an alternative solution to Question 1.2. in fact, the inner automorphism group of a LTWC-loop is generated by just one inner mapping.

3 Main Results

3.1 RTWC-Loop and LTWC-Loop

In a LTWC (RTWC)-loop, the following are equivalent:

1. Flexibility. 2. RAP (LAP). 3. RIP (LIP).

Hence, any LTWC (RTWC)-loop which obeys any of them is an extra loop. Any LTWC (RTWC)-loop is a Moufang loop if and only if it is an extra loop. In a LTWC-loop G, $N_\mu(G) = N_\lambda(G)$. While in a RTWC-loop G, $N_\mu(G) = N_\rho(G)$.

These observations necessitate the introduction of the properties in Definition 3.1 for investigation in addition to the fact that it was shown in [8] that a LTWC-loop is power associative.

Definition 3.1. *A power associative loop G is said to obey*

1. *LAP_n-property if $x \cdot x^n y = x^{n+1} y$ where $n \in \mathbb{Z}$, $\forall\ x, y \in G$.*
2. *LAP^n-property if $x^n \cdot xy = x^{n+1} y$ where $n \in \mathbb{Z}$, $\forall\ x, y \in G$.*
3. *$FLEX^n$-property if $x^n \cdot yx = x^n y \cdot x$ where $n \in \mathbb{Z}$, $\forall\ x, y \in G$.*

4. *$FLEX_n$-property if $x \cdot yx^n = xy \cdot x^n$ where $n \in \mathbb{Z}$, $\forall\ x, y \in G$.*

5. *$FLEX_{(2,n+1)}$-property if $x^2y \cdot x^n = x^2 \cdot yx^n$ where $n \in \mathbb{Z}$, $\forall\ x, y \in G$.*

6. *RAP^n-property if $yx \cdot x^n = yx^{n+1}$ where $n \in \mathbb{Z}$, $\forall\ x, y \in G$.*

7. *RAP_n-property if $yx^n \cdot x = yx^{n+1}$ where $n \in \mathbb{Z}$, $\forall\ x, y \in G$.*

Remark 3.2. *In any power associative loop: LIP is equivalent to the property LAP^{-1} or LAP_{-1}; RIP is equivalent to the property RAP^{-1} or RAP_{-1},*

Theorem 3.3. *Let G be a LTWC (RTWC)-loop. G is a Frute loop if and only if G is a crazy loop.*

Proof. Let G be a LTWC-loop and suppose G is also a crazy (A4C3) loop, i.e., $(x \cdot xy)z = yx \cdot xz$. Put $z \mapsto xz$ in the crazy identity, then we have

$$(x \cdot xy) \cdot xz = yx \cdot (x \cdot xz) = x((x \cdot yx)z) \Rightarrow x(xy \cdot z) = y(x \cdot xz)$$

which is A2C1. So, G is a Frute loop.

Conversely, let G be a LTWC-loop and suppose also that G is also a Frute (A2C1) loop. Put $y = y/x$ in the LTWC identity, we have $x(xy \cdot z) = (x(x(y/x))) \cdot xz$, and so since G is also Frute loop (A2C1); $y(x \cdot xz) = (x(x(y/x))) \cdot xz$ Thus, G is a crazy loop.

□

Remark 3.4. *Theorem 3.3(3.3) is nontrivial because by (Theorem 2.6 (2) of [14]), in a Frute loop, the property $x^{-1}yx = xyx^{-1}$ (which is equivalent to the centrum square property in a diassociative loop) is satisfied but not generally true in Moufang, extra loop or group. Hence, it distinguishes a Frute loop from a Moufang, extra loop and group. In (Theorem 2.2 of [14]), this property is necessary and sufficiency for a group to be a Frute loop and vice versa. But may not be necessary and sufficiency for a Frute loop to be a crazy loop (which is a group and has the property itself).*

Theorem 3.5. *Let G be a LTWC-loop.*

1. *$x \cdot x^n y = x^n \cdot xy\ \forall x, y \in G,\ n \in \mathbb{Z}$.*

2. *G is left power alternative $\Leftrightarrow$ G has the LAP_n-property for all $n \in \mathbb{Z} \Leftrightarrow$ G has the LAP^n-property for all $n \in \mathbb{Z}$.*

3. *For any fixed $n \in \mathbb{Z}$, G has the LAP^n-property $\Leftrightarrow$ G has the LAP_n-property.*

4. *G has the $FLEX_n$, LAP^1 and $FLEX_{(2,n+1)}$ properties implies that G has the RAP^n property for any fixed $n \in \mathbb{Z}\backslash\{1\}$.*

5. *For any fixed $n \in \mathbb{Z}$, the LPA^{n+1} and LPA^n properties imply the LAP^n and LAP_n properties.*

6. *Left power alternative property implies the LAP^n and LAP_n properties for all $n \in \mathbb{Z}$.*

7. *LAP^n or LAP_n and LPA^n properties imply the LPA^{n+1} property for any fixed $n \in \mathbb{Z}$.*

Proof. 1. We claim that $x \cdot x^{n-1}y = x^{n-1} \cdot xy$ for all $x, y \in G$, $n \in \mathbb{Z}$.

$$\text{For } n = 1,\ xy = xy \text{ (trivial).}$$
$$\text{For } n = 2,\ x \cdot xy = x \cdot xy.$$
$$\text{For } n = 3,\ x \cdot x^2y = x^2 \cdot xy.$$

We assume that $x \cdot x^{n-1}y = x^{n-1} \cdot xy$ for all $n \leq k$, that is $x \cdot x^{k-1}y = x^{k-1} \cdot xy$.

Put $y = x^{k-2}$ in LTWC identity to get: $x \cdot (x \cdot x^{k-2}x)z = (x \cdot xx^{k-2}) \cdot xz \Rightarrow x \cdot (xx^{k-1})z = (xx^{k-1}) \cdot xz \Rightarrow x \cdot x^kz = x^k \cdot xz \Rightarrow x \cdot x^{(k-1)+1}z = x^{(k-1)+1} \cdot xz$. Replace $k-1$ by n to get $x \cdot x^{n+1}z = x^{n+1} \cdot xz$. So, $x \cdot x^nz = x^n \cdot xz\ \forall\ n \in \mathbb{N}$.

Put $y = x^{-k-2}$ in LTWC to get: $x \cdot (x \cdot x^{-k-2}x)z = (x \cdot xx^{-k-2}) \cdot xz \Rightarrow x \cdot (xx^{-k-1})z = (xx^{-k-1}) \cdot xz \Rightarrow x \cdot x^{-k}z = x^{-k} \cdot xz$. So, $x \cdot x^ky = x^k \cdot xy$, $\forall\ k \in \mathbb{Z}^-$. Therefore, $x \cdot x^ny = x^n \cdot xy\ \forall x, y \in G$, $n \in \mathbb{Z}$.

2. G is left power alternative $\Leftrightarrow \underbrace{x(\cdots(x(x}_{n+1-\text{times}}y))\cdots) = x^{n+1}y \ \forall x, y \in G$ and $n \in \mathbb{Z}$. With 1, it will be observed that:

$$\underbrace{x(\cdots(x(x}_{n-\text{times}}y))\cdots) = x^n y$$

implies $x \cdot x^{n-1}y = x^n y$ and $x^{n-1} \cdot xy = x^n y$ for all $x, y \in G$ and $n \in \mathbb{Z}$. Likewise,

$$x \cdot x^{n-1}y = x^n y \text{ and } x^{n-1} \cdot xy = x^n y \text{ when } n = 1, 2, 3, \cdots, i$$

imply

$$\underbrace{x(\cdots(x(x}_{n-\text{times}}y))\cdots) = x^n y \text{ when } n = i + 1$$

for all $x, y \in G$ and $i \in \mathbb{Z}$.

3. Use 1.

4. Put $z = x$ in LTWC to get $x[(x \cdot yx)x] = (x \cdot xy)x^2$. Then,

$$\text{FLEX}_1 \text{ property in } G \Rightarrow x[x(yx \cdot x)] = (x \cdot xy)x^2 \underset{\text{property}}{\overset{\text{LAP}^1}{\Longrightarrow}}$$

$$x^2(yx \cdot x) = x^2y \cdot x^2 \underset{\text{property}}{\overset{\text{FLEX}_{(2,2)}}{\Longrightarrow}} x^2(yx \cdot x) = x^2 \cdot yx^2 \Rightarrow$$

$$yx \cdot x = yx^2 \Rightarrow \text{RAP}^1 \text{ property.}$$

Put $z = x^2$ in LTWC to get $x[(x \cdot yx)x^2] = (x \cdot xy)x^3$. Then,

$$\text{FLEX}_2 \text{ property in } G \Rightarrow x[x \cdot (yx)x^2] = (x \cdot xy)x^3 \underset{\text{property}}{\overset{\text{LAP}^1}{\Longrightarrow}}$$

$$x^2(yx \cdot x^2) = x^2y \cdot x^3 \underset{\text{property}}{\overset{\text{FLEX}_{(2,3)}}{\Longrightarrow}} x^2(yx \cdot x^2) = x^2 \cdot yx^3 \Rightarrow$$

$$yx \cdot x^2 = yx^3 \Rightarrow \text{RAP}^2 \text{ property.}$$

Put $z = x^3$ in LTWC to get $x[(x \cdot yx)x^3] = (x \cdot xy)x^4$. Then,

$$\text{FLEX}_3 \text{ property in } G \Rightarrow x[x \cdot (yx)x^3] = (x \cdot xy)x^4 \underset{\text{property}}{\overset{\text{LAP}^1}{\Longrightarrow}}$$

$$x^2(yx \cdot x^3) = x^2y \cdot x^4 \underset{\text{property}}{\overset{\text{FLEX}_{(2,4)}}{\Longrightarrow}} x^2(yx \cdot x^3) = x^2 \cdot yx^4 \Rightarrow$$

$$yx \cdot x^3 = yx^4 \Rightarrow \text{RAP}^3 \text{ property.}$$

Put $z = x^n$ in LTWC to get $x[(x \cdot yx)x^n] = (x \cdot xy)x^{n+1}$. Then,

$$\text{FLEX}_n \text{ property in } G \Rightarrow x[x \cdot (yx)x^n] = (x \cdot xy)x^{n+1} \underset{\text{property}}{\overset{\text{LAP}^1}{\Longrightarrow}}$$

$$x^2(yx \cdot x^n) = x^2y \cdot x^{n+1} \underset{\text{property}}{\overset{\text{FLEX}_{(2,n+1)}}{\Longrightarrow}} x^2(yx \cdot x^n) = x^2 \cdot yx^{n+1}$$

$$\Rightarrow yx \cdot x^n = yx^{n+1} \Rightarrow \text{RAP}^n \text{ property.}$$

So, RAP^n property is true for all $n \in \mathbb{N}$.

Put $z = x^{-n}$ in LTWC to get $x[(x \cdot yx)x^{-n}] = (x \cdot xy)x^{-n+1}$. Then,

$$\text{FLEX}_{-n} \text{ property in } G \Rightarrow x[x \cdot (yx)x^{-n}] = (x \cdot xy)x^{-n+1} \underset{\text{property}}{\overset{\text{LAP}^1}{\Longrightarrow}}$$

$$x^2(yx \cdot x^{-n}) = x^2y \cdot x^{-n+1} \underset{\text{property}}{\overset{\text{FLEX}_{(2,-n+1)}}{\Longrightarrow}} x^2(yx \cdot x^{-n}) = x^2 \cdot yx^{-n+1} \Rightarrow$$

$$yx \cdot x^{-n} = yx^{-n+1} \Rightarrow yx \cdot x^n = yx^{n+1} \Rightarrow \text{RAP}^{-n} \text{ property.}$$

So, RAP^n property is true for all $n \in \mathbb{Z}^-$. Therefore, RAP^n property is true for all $n \in \mathbb{Z}$.

5. For any fixed $n \in \mathbb{Z}$, LPA^n-property hold if and only if $L_x^n = L_{x^n}$ for all $x \in G$. Thus, $L_x^{n+1} = L_x^n L_x = L_{x^n} L_x = L_x L_{x^n}$. So, LAP^n and LAP_n properties hold.

6. This follows by 5.

7. This is similar to 5.

□

3.2 Total inner mapping group of LTWC loop

We shall be discussing some properties of translations of LTWC-loop in this subsection for application in the next subsection.

Theorem 3.6. *In a LTWC loop $(G,\cdot)$ and for each $x,y \in G$:*

1. $R_x^m = L_x^{2n} R_x^m L_x^{-2n}$ *or* $[R_x^m, L_x^{2n}] = I$ *for any* $m,n \in \mathbb{Z}$.
2. $R_{x^n} = L_x^{-2} T_x^{n-2} (R_x L_x)^2 L_x^{n-2}$ *for any* $n \in \mathbb{Z}$.
3. $R_{x^n} = T_x^{n-1} R_x L_x^{n-1}$ *for any* $n \in \mathbb{Z}$.
4. $R_{(x,y)} = T_{(x,y)} L_x^{-1} S_x L_x = T_{(x,y)} T_x^{-1}$ *where* $S_x = R_x^{-1} L_x$ *and* $T_{(x,y)} = R_x R_y L_x^{-1} R_y^{-1}$ *are elements of* $Inn(G)$. $Inn(G) = \langle T_{(x,y)} \mid x,y \in G \rangle$ *and* $TInn(G,\cdot) = \langle T_{(x,y)}, L_{(x,y)}, M_{(x,y)}, U_x \mid x,y \in G \rangle =$ $\langle T_{(x,y)}, L_{(x,y)}, P_{(x,y)}, V_x \mid x,y \in G \rangle$.

Proof. G is a LTWC-loop if and only if for all $x,y \in G$,

$$R_x L_x R_z L_x = L_x^2 R_{xz} \tag{1}$$

A LTWC-loop G is an LCC-loop, and so, from the LCC-identity, for all $x,y \in G$,

$$R_y L_x = L_x R_x^{-1} R_{xy} \tag{2}$$

For all $x,y \in G$, the LCC-identity is equivalent to

$$L_x^{-1} L_y L_x = L_{xy/x} \tag{3}$$

Since G is a LTWC-loop, then $x(x^2 z) = x^2 \cdot xz$ for all $x,z \in G$. So, we have

$$L_x L_{x^2} = L_{x^2} L_x \quad \text{or} \quad [L_{x^2}, L_x] = I \tag{4}$$

1. Put $z = e$ in (1) to get $R_x L_x^2 = L_x^2 R_x$. So, $R_x = L_x^2 R_x L_x^{-2} \Rightarrow R_x = L_x^{2n} R_x L_x^{-2n}$ for all $n \in \mathbb{Z}$. Also, $R_x^2 = L_x^2 R_x^2 L_x^{-2}$, $R_x^3 = L_x^2 R_x^3 L_x^{-2}, \cdots, R_x^m = L_x^2 R_x^m L_x^{-2}$. Then $R_x^m = L_x^{2n} R_x^m L_x^{-2n} \Leftrightarrow R_x^m L_x^{2n} = L_x^{2n} R_x^m \; \forall n,m \in \mathbb{Z} \Leftrightarrow [R_x^m, L_x^{2n}] = I \; \forall \; n,m \in \mathbb{Z}$.

2. Put $z = e$ in (1) to get $R_x L_x^2 = L_x^2 R_x \Rightarrow R_x = L_x^{-2} R_x L_x^2$
$= L_x^{-2}(R_x L_x^{-1})^{-1}(R_x L_x)^2 L_x^{-1}$.
Put $z = x$ in (1), then $R_x L_x R_x L_x = L_x^2 R_{x^2} \Rightarrow R_{x^2} = L_x^{-2} R_x L_x R_x L_x =$

$L_x^{-2}(R_x L_x^{-1})^0 (R_x L_x)^2 L_x^0$.
Put $z = x^2$ in (1), then $R_x L_x R_{x^2} L_x = L_x^2 R_{x^3} \Rightarrow R_{x^3} = L_x^{-2} R_x L_x R_{x^2} L_x =$
$L_x^{-2} R_x L_x L_x^{-2} R_x L_x R_x L_x L_x$
Put $z = x^3$ in (1) to get $R_x L_x R_{x^3} L_x = L_x^2 R_{x^4} \Rightarrow R_{x^4} = L_x^{-2} R_x L_x R_{x^3} L_x =$
$L_x^{-2} R_x L_x L_x^{-2} R_x L_x^{-1} (R_x L_x)^2 L_x L_x = L_x^{-2} R_x L_x^{-1} R_x L_x^{-1} (R_x L_x)^2 L_x^2 =$
$L_x^{-2}(R_x L_x^{-1})^2 (R_x L_x)^2 L_x^2$.
Put $z = x^4$ in (1), then $R_x L_x R_{x^4} L_x = L_x R_{x^5} \Rightarrow R_{x^5} = L_x^{-2} R_x L_x R_{x^4} L_x =$
$L_x^{-2} R_x L_x L_x^{-2} (R_x L_x^{-1})^2 (R_x L_x)^2 L_x^3 = L_x^{-2}(R_x L_x^{-1})^3 (R_x L_x)^2 L_x^3$.
Put $z = x^5$ in (1) to get $R_x L_x R_{x^5} L_x = L_x^2 R_{x^6} \Rightarrow R_{x^6} = L_x^{-2} R_x L_x R_{x^5} L_x$
$= L_x^{-2} R_x L_x L_x^{-2} (R_x L_x^{-1})^3 (R_x L_x)^2 L_x^3 L_x = L_x^{-2}(R_x L_x^{-1})^4 (R_x L_x)^2 L_x^4$.

We claim that $R_{x^n} = L_x^{-2}(R_x L_x^{-1})^{n-2}(R_x L_x)^2 L_x^{n-2} \ \forall\ n \in \mathbb{N}$. When
$n = 1, 2, 3, 4, 5, 6$ (true) going by computations above.
Let $R_{x^k} = L_x^{-2}(R_x L_x^{-1})^{k-2}(R_x L_x)^2 L_x^{k-2}$, for $n = k$.
Put $z = x^k$ in (1) to get $R_x L_x R_{x^k} L_x = L_x R_{x^{k+1}}$.

$= L_x^{-2} R_x L_x L_x^{-2} (R_x L_x^{-1})^{k-2} (R_x L_x)^2 L_x^{k-2} L_x$
$= L_x^{-2}(R_x L_x^{-1})^{k-1} (R_x L_x)^2 L_x^{k-1}$.

Replace $k+1$ with n to get $R_{x^n} = L_x^{-2}(R_x L_x^{-1})^{n-2}(R_x L_x)^2 L_x^{n-2}$.
Put $z = x^{-1}$ in (1), then $R_x L_x R_{x^{-1}} L_x = L_x^2 R_e \Rightarrow R_{x^{-1}} = L_x^{-1} R_x^{-1} L_x^2 L_x^{-1}$
$= L_x^{-1} R_x^{-1} L_x = L_x^{-2}(R_x L_x^{-1})^{-3}(R_x L_x)^2 L_x^{-3}$.
Put $z = x^{-2}$ in (1), then $R_x L_x R_{x^{-2}} L_x = L_x R_{x^{-1}} \Rightarrow R_{x^{-2}} = L_x^{-1} R_x^{-1} L_x^2 R_{x^{-1}} L_x^{-1}$
$= L_x^{-1} R_x^{-1} L_x^2 \left(L_x^{-2}(R_x L_x^{-1})^{-3}(R_x L_x)^2 L_x^{-3}\right) L_x^{-1} = L_x^{-2}(R_x L_x^{-1})^{-4}(R_x L_x)^2 L_x^{-4}$.
We claim that $R_{x^n} = L_x^{-2}(R_x L_x^{-1})^{n-2}(R_x L_x)^2 L_x^{n-2} \ \forall\ n \in \mathbb{Z}^-$. When
$n = -1, -2$ (true) going by computations above.
Assume that $R_{x^{-k}} = L_x^{-2}(R_x L_x^{-1})^{-k-2}(R_x L_x)^2 L_x^{-k-2}$.

Put $z = x^{-k-1}$ in (1) to get $R_x L_x R_{x^{-k-1}} L_x = L_x^2 R_{xx^{-k-1}} \Rightarrow R_x L_x R_{x^{-k-1}} L_x = L_x^2 R_{x^{-k}}$.

$$\begin{aligned} R_{x^{-(k+1)}} &= L_x^{-1} R_x^{-1} L_x^2 R_{x^{-k}} = L_x^{-1} R_x^{-1} L_x^2 (L_x^{-2} (R_x L_x^{-1})^{-k-2} (R_x L_x)^2 L_x^{-k-2} L_x^{-1} \\ &= L_x^{-1} R_x^{-1} (R_x L_x^{-1})^{-k-2} (R_x L_x)^2 L_x^{-k-3} \\ &= L_x^{-2} (R_x L_x^{-1})^{-1} (R_x L_x^{-1})^{-k-2} (R_x L_x)^2 L_x^{-k-3} \\ &= L_x^{-2} (R_x L_x^{-1})^{-k-3} (R_x L_x)^2 L_x^{-k-3} \\ R_{x^{-(k+1)}} &= L_x^{-2} (R_x L_x^{-1})^{-(k+1)-2} (R_x L_x)^2 L_x^{-(k+1)-2}. \end{aligned}$$

Replace $k+1$ with n to get $R_{x^{-n}} = L_x^{-2} (R_x L_x^{-1})^{-n-2} (R_x L_x)^2 L_x^{-n-2}$.

Therefore, $R_{x^n} = L_x^{-2} T_x^{n-2} (R_x L_x)^2 L_x^{n-2} \; \forall \; n \in \mathbb{Z}$.

3. Use (2) in the similar way in which (1) was used to prove 2.

4. Using (1), we get

$$\begin{aligned} R_{(x,y)} =& R_x R_y R_{xy}^{-1} = R_x R_y \left(L_x^{-2} R_x L_x R_y L_x \right)^{-1} = R_x R_y L_x^{-1} R_y^{-1} L_x^{-1} R_x^{-1} L_x^2 \\ =& T_{(x,y)} L_x^{-1} S_x L_x. \end{aligned}$$

Using (2), we get

$$\begin{aligned} R_{(x,y)} =& R_x R_y R_{xy}^{-1} = R_x R_y \left(R_x L_x^{-1} R_y L_x \right)^{-1} = R_x R_y L_x^{-1} R_y^{-1} L_x R_x^{-1} \\ =& T_{(x,y)} T_x^{-1}. \end{aligned}$$

□

Remark 3.7. *By Theorem 3.6(4), LTWC-loop is thus a class of loop which falls into the Syrbu and Grecu (2019) first answer to a question of Stanovský and Vojtěchovský (2014)-minimal generating set for the total inner mapping group of a loop. In addition, the class of LTWC-loop also provides an alternative solution to 2014 question and in fact, the inner automorphism group of a LTWC-loop is generated by just one inner mapping.*

Corollary 3.8. *Let G be a LTWC-loop. For any $x \in G$, $n \in \mathbb{N}$:*

1. $|x| = n \Leftrightarrow T_x^n(R_xL_x)^2L_x^n = T_x^2L_x^4 \Leftrightarrow T_x^nL_x^n = I$.

2. $|T_x| = n \Leftrightarrow L_x^2R_{x^{n+2}} = (R_xL_x)^2L_x^n \Leftrightarrow R_{x^{n+1}} = R_xL_x^n$.

3. $|L_x| = n \Leftrightarrow L_x^2R_{x^{n+2}} = T_x^n(R_xL_x)^2 \Leftrightarrow R_{x^{n+1}} = T_x^nR_x$.

4. *G has the RAP^{n+1}- property if and only if* $L_x^2R_xR_{x^{n+1}} = T_x^n(R_xL_x)^2L_x^n$; *$G$ has RAP if and only if* $(x \cdot xy)x \cdot x = x \cdot (x \cdot yx)x$.

5. *G has the RAP^n-property if and only if* $R_xR_{x^n} = T_x^nR_xL_x^n$; *$G$ has RAP if and only if* $T_xR_x = R_xT_x$.

6. *G has the RAP_{n+1}-property if and only if* $L_x^2R_{x^{n+1}}R_x = T_x^n(R_xL_x)^2L_x^n$.

7. *G has the RAP_n-property if and only if* $R_{x^n}R_x = T_x^nR_xL_x^n$.

8. *G has the RPA^{n+2}- property if and only if* $L_x^2R_{x^{n+2}} = T_x^n(R_xL_x)^2L_x^n$

9. *G has the RPA^{n+1}-property if and only if* $R_{x^{n+1}} = T_x^nR_xL_x^n$.

10. *G has the LPA^n-property if and only if* $L_x^2R_{x^{n+2}} = T_x^n(R_xL_x)^2L_{x^n}$; *$G$ has LAP if and only if* $(x \cdot xy)x^4 = x^2 \cdot x[x[x\backslash(x\backslash(yx) \cdot x) \cdot x] \cdot x]$.

11. *G has the LPA^n-property if and only if* $R_{x^{n+1}} = T_x^nR_xL_{x^n}$; *$G$ has LAP if and only if* $yx^3 = x^2[x\backslash(yx) \cdot x]$.

Proof. We shall use Theorem 3.6.

1. Going by 2, $|x| = n \Leftrightarrow L_x^{-2}T_x^{n-2}(R_xL_x)^2L_x^{n-2} = I \Leftrightarrow T_x^{n-2}(R_xL_x)^2L_x^n = L_x^2L_x^2 \Leftrightarrow T_x^n(R_xL_x)^2L_x^n = T_x^2L_x^4$. Also, going by 3, $|x| = n \Leftrightarrow T_x^{n-1}R_xL_x^{n-1} = I \Leftrightarrow T_x^{n-1}R_xL_x^{-1}L_x^n = I \Leftrightarrow T_x^nL_x^n = I$.

2. Similarly, use 2 and 3.

3. Similarly, use 2 and 3.

The proofs of 4 to 11 follow by using 2 and 3. □

Remark 3.9. *In Corollary 3.8, we have established necessary and sufficient condition for a LTWC-loop to have LAP (RAP) or to be left (right) power alternative.*

Remark 3.10. *Going by Corollary 3.8(2), with $n = 1$, a LTWC-loop G is commutative if and only if $x \cdot yx = yx^2$ or $x[x \cdot (x \cdot yx)x] = (x \cdot xy)x^3$ for all $x, y \in G$.*

3.3 Construction of LTWC-loop

We shall be using some of the properties of the translations of LTWC-loop derived in previous subsection for construction. We adopt the following notations for such in a loop $(G, \cdot)$:

$$\prod_{\rho}(G,\cdot) = \prod_{\rho}(G) = \{R_a \mid a \in G\} = \{R(a) \mid a \in G\} \quad \text{and}$$

$$\prod_{\lambda}(G,\cdot) = \prod_{\lambda}(G) = \{L_a \mid a \in G\} = \{L(a) \mid a \in G\}.$$

Designate the elements of G by $x_1, \cdots, x_n$ and R_{x_j} by $R_j = R(j)$, and assume henceforth that R_1 is the identity permutation. Then $R_2, \cdots, R_n,,$ are permutations on n letters and $R(i) = (1\ i\ \cdots)$. Conversely if $(1\ i\ \cdots)$ is in Π_ρ, then $(1\ i\ \cdots) = R_i = R(i)$.

Theorem 3.11. *Let G be a loop.*

1. *G is a LTWC-loop if and only if for all $\alpha \in \prod_\rho$, for some $\beta \in \prod_\lambda$, and for some $\alpha' \in \prod_\rho$, $\beta^{-2}\alpha'\beta\alpha\beta \in \prod_\rho$.*
2. *If G is a LTWC-loop, then*
   - *(a) $\beta, \beta' \in \prod_\lambda \Rightarrow \beta^{-1}\beta'\beta \in \prod_\lambda$.*
   - *(b) for some $\alpha \in \prod_\lambda$ and $\beta \in \prod_\rho$, $\beta\alpha^2 = \alpha^2\beta$.*
   - *(c) for some $\alpha \in \prod_\lambda$ and $\beta \in \prod_\rho$, $\alpha^{-2}(\beta\alpha^{-1})^{n-2}(\beta\alpha)^2\alpha^{n-2} \in \prod_\rho$ for all $n \in \mathbb{Z}$.*
   - *(d) for some $\alpha \in \prod_\lambda$ and $\beta \in \prod_\rho$, $(\beta\alpha^{-1})^{n-1}\beta\alpha^{n-1} \in \prod_\rho$ for all $n \in \mathbb{Z}$.*
   - *(e) for some $\alpha \in \prod_\lambda$ and $\beta \in \prod_\rho$, $\beta^m\alpha^{2n} = \alpha^{2n}\beta^m\ \forall\ m, n \in \mathbb{Z}$.*
   - *(f) for some $\beta, \beta' \in \prod_\lambda$, $\beta\beta' = \beta'\beta$.*

(g) *for all* $\alpha \in \prod_\rho$, *for some* $\beta \in \prod_\lambda$, *and for some* $\alpha' \in \prod_\rho$, $\alpha'\beta^{-1}\alpha\beta \in \prod_\lambda$.

Proof. Use Theorem 3.6, including (1), (2), (3) and (4). □

Theorem 3.12. *Let G be a LTWC-loop.*

1. *If there exists $x \in G$ such that $G = \langle x \rangle$, then for some $\beta \in \prod_\rho$,*

$$\prod_\rho(G) = \left\{ \alpha^{-2} \left(\beta\alpha^{-1}\right)^{n-2} (\beta\alpha)^2 \alpha^{n-2} \mid \text{for some } \alpha \in \prod_\lambda,\ n \in \mathbb{Z} \right\}$$

and G is a group.

2. *If there exist $\{x_i\}_{i=1}^{r\geq 2} \subset G$ such that $\bigcup_{i=1}^{r} \langle x_i \rangle = G$, then for some $\beta \in \prod_\rho$,*

$$\prod_\rho(G) = \left\{ \alpha^{-2} \left(\beta\alpha^{-1}\right)^{n-2} (\beta\alpha)^2 \alpha^{n-2} \mid \text{for some } \alpha \in \prod_\lambda,\ n \in \mathbb{Z} \right\}.$$

3. *If there exists $x \in G$ such that $G = \langle x \rangle$, then for some $\beta \in \prod_\rho$,*

$$\prod_\rho(G) = \left\{ \left(\beta\alpha^{-1}\right)^{n-1} \beta\alpha^{n-1} \mid \text{for some } \alpha \in \prod_\lambda,\ n \in \mathbb{Z} \right\}$$

and G is a group.

4. *If there exist $\{x_i\}_{i=1}^{r\geq 2} \subset G$ such that $\bigcup_{i=1}^{r} \langle x_i \rangle = G$, then for some $\beta \in \prod_\rho$,*

$$\prod_\rho(G) = \left\{ \left(\beta\alpha^{-1}\right)^{n-1} \beta\alpha^{n-1} \mid \text{for some } \alpha \in \prod_\lambda,\ n \in \mathbb{Z} \right\}.$$

5. *If there exist* $\{x_i\}_{i=1}^{r\geq 2} \subset G$ *such that* $\bigcup_{i,j=1}^{r} \langle x_i, x_j\rangle = G$, *then given any* $\alpha \in \prod_\rho$ *and for some* $\alpha' \in \prod_\rho$, $\prod_\rho(G) = \left\{\beta^{-2}\alpha'\beta\alpha\beta \mid \text{for some } \beta \in \prod_\lambda\right\}$.

6. *If there exist* $\{x_i\}_{i=1}^{r\geq 2} \subset G$ *such that* $\bigcup_{i,j=1}^{r} \langle x_i, x_j\rangle = G$, *then given any* $\alpha \in \prod_\rho$ *and for some* $\alpha' \in \prod_\rho$, $\prod_\rho(G) = \left\{\alpha'\beta^{-1}\alpha\beta \mid \text{for some } \beta \in \prod_\lambda\right\}$.

Proof. Let G be a LTWC-loop. We shall use Theorem 3.6 and Theorem 3.11.

1. If there exists $x \in G$ such that $G = \langle x\rangle$, then $R_{x^n} = L_x^{-2}T_x^{n-2}(R_xL_x)^2L_x^{n-2}$ for any $n \in \mathbb{Z}$ implies $\prod_\rho(G) = \left\{\alpha^{-2}\left(\beta\alpha^{-1}\right)^{n-2}(\beta\alpha)^2\alpha^{n-2} \mid \text{for some } \alpha \in \prod_\lambda,\ n \in \mathbb{Z}\right\}$ for some $\beta \in \prod_\rho$. A LTWC-loop is power associative, thus, $G = \langle x\rangle$ is a group.

2. If there exist $\{x_i\}_{i=1}^{r\geq 2} \subset G$ such that $\bigcup_{i=1}^{r}\langle x_i\rangle = G$, then $R_{x^n} = L_x^{-2}T_x^{n-2}(R_xL_x)^2L_x^{n-2}$ for any $n \in \mathbb{Z}$ implies $\prod_\rho(G) = \left\{\alpha^{-2}\left(\beta\alpha^{-1}\right)^{n-2}(\beta\alpha)^2\alpha^{n-2} \mid \text{for some } \alpha \in \prod_\lambda,\ n \in \mathbb{Z}\right\}$ for some $\beta \in \prod_\rho$.

3. This is similar to 1 using $R_{x^n} = T_x^{n-1}R_xL_x^{n-1}$ for any $n \in \mathbb{Z}$ in a LTWC-loop.

4. This is similar to 2 using $R_{x^n} = T_x^{n-1}R_xL_x^{n-1}$ for any $n \in \mathbb{Z}$ in a LTWC-loop.

5. This is similar to 2 using $R_{xy} = L_x^{-2}R_xL_xR_yL_x$ with $x, y \in \{x_i\}_{i=1}^{r\geq 2}$ in a LTWC-loop.

6. This is similar to 2 using $R_{xy} = R_x L_x^{-1} R_y L_x$ with $x, y \in \{x_i\}_{i=1}^{r\geq 2}$ in a LTWC-loop.

□

Example 3.13. *Let G be a LTWC-loop of order 8. Given*

$$\beta = \beta_7 = (1\ 6\ 4\ 7)(2\ 8)(3\ 5) = R(6),\ \beta' = \beta_8 = (1\ 2)(3\ 4)(5\ 7\ 8\ 6) = R(2),$$
$$\alpha = (1\ 6\ 4\ 7)(2\ 5\ 3\ 8) = L(6),\ \alpha' = (1\ 2)(3\ 4)(5\ 7)(6\ 8) = L(2) \in \prod_{\lambda}(G).$$

We shall now apply Theorem 3.11.

When, $n = 2$, $\beta_1 = \left(\beta\alpha^{-1}\right)^{2-1}\beta\alpha^{2-1} = (1\ 4)(2\ 3)(5\ 8)(6\ 7) = R(4) \in \prod_{\rho}(G).$

When, $n = 3$, $\beta_2 = \left(\beta\alpha^{-1}\right)^{3-1}\beta\alpha^{3-1} = (1\ 7\ 4\ 6)(2\ 5)(3\ 8) = R(7) \in \prod_{\rho}(G).$

When, $n = 4$, $\beta_3 = \left(\beta\alpha^{-1}\right)^{4-1}\beta\alpha^{4-1} = (1)(2)(3)(4)(5)(6)(7)(8) = R(1) \in \prod_{\rho}(G).$

$$\beta_4 = \beta\alpha^{-1}\beta'\alpha = (1\ 5)(2\ 7)(3\ 6)(4\ 8) = R(5) \in \prod_{\rho}(G).$$

$$\beta_5 = \beta'\alpha'^{-1}\beta\alpha' = (1\ 8)(2\ 6)(3\ 7)(4\ 5) = R(8) \in \prod_{\rho}(G).$$

$$\beta_6 = \beta'\alpha'^{-1}\beta_1\alpha' = (1\ 3)(2\ 4)(5\ 6\ 8\ 7) = R(3) \in \prod_{\rho}(G).$$

Then, $\prod_\rho(G) = \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8\}$. *Consequently, the multiplication table of G is presented in Table 1.*

4 Concluding Remarks

The generalized RAP, LAP and FLEX properties in Definition 3.1 can be viewed as the 'measure' of difference between a LTWC (RTWC)-loop being power associative and not being diassociative viz-a-vis the results in Theorem 3.5 and Corollary 3.8.

$\cdot$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	4	3	7	8	5	6
3	3	4	1	2	6	5	8	7
4	4	3	2	1	8	7	6	5
5	5	7	6	8	1	3	2	4
6	6	5	8	7	3	4	1	2
7	7	8	5	6	2	1	4	3
8	8	6	7	5	4	2	3	1

Table 1: A LTWC-loop $(G, \cdot)$ of order 8

In literature, there are some varieties of loops of first Bol-Moufang types which have RAP, LAP, power right (left) alternative properties (e.g. LC, RC, central [1, 9], left/right Bol) which are power associative but not necessarily diassociative. There are also some varieties of loops first Bol-Moufang type which have RAP, LAP, power right (left) alternative properties, flexibility (e.g. Moufang, extra) which are both power associative and diassociative. What can then be said about varieties of loops of second Bol-Moufang types such as LTWC (RTWC)-loop ?

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THE GEOMETRY OF FLUCTUATIONS, I (NONEQUILIBRIUM STATISTICAL MECHANICS)

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Abstract

We present a formulation of non-linear non-equilibrium statistical mechanics in terms of the stochastic differential geometry associated to the Riemann-Cartan-Weyl (RCW) geometries with Weyl-torsion and their diffusion processes. We give the derivation of the covariant Fokker-Planck master. The Weyl torsion plays the fundamental role of describing the first moment (incorporating also the fluctuations due to the second moment) of the stochastic processes and geometrically has its trivial term produced by the Lie-isotopic orthogonal group unit plus two electromagnetic potentials which couple to the invariant density produced by the isotopic unit. We analize the problem of detailed balance -as a formalization of microscopic reversibility- and its relation with the probability vector field given by the electromagnetic potentials terms in the Weyl torsion. These potentials, which appear in the usual setting of the problem of coupling of the electromagnetic potential to torsion, are found to violate detailed balance and thus give rise to microscopical irreversibility. Yet, we do not require the existance of a hamiltonean function. We discuss the relation between these stochastic processes and a non-covariant derivation of the Liouville equation of classical mechanics. We introduce the kinetic potential and its image, of non-linear non-equilibrium statistical mechanics, and show that the Weyl torsion plays a fundamental role in warranting that it is a convex function. We finally prove that the non-linear non-equilibrium H-theorem and the fluctuation-dissipation equations are based in the trace-torsion vector field, and that these equations give rise to a local arrow of time.

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INTRODUCTION

The application of differential geometry to the description of physical phenomenae is probably one of the most coherent attempts of this century in the development of a unified perspective. Undoubtedly, Einstein's relativity set the course for this development, and it was the work of the fifties in the renaissance of gauge theories of the fundamental interactions, that established that geometry in its relation to group symmetries yields a mighty methodology. It was the time of the seventies and further which called attention on the symplectic geometry structure of classical mechanics, and contemporarily this geometry has been extended to encompass quantum field theory.

Yet, all these efforts concentrated mainly in describing conservative systems, disregarding the description of dissipative systems. Yet, to mention an exception, the problem of integrability of classical dynamical systems opened a new view of nature in which complexity and stochasticity appeared in a realm which could hardly be suspected to exist. Thus, notions of statistical mechanics, such as ergodicity, came to the center of classical and quantum mechanics.

Yet, probably one of the basic lessons provided by the Symplectic Geometry structure of physics, as constructed -for instance- by Souriau [23,24,11,24] is that notions such as hamiltonean functions, are not necessary even for the description of classical conservative systems. Quite different, remarkably, is the lagrangian formulation of classical systems, as elaborated in the pioneering works by Santilli [5].

Santilli's research [6,7,21] progressed in several complementary aspects of dissipative systems, the algebraic, geometrical and analytic structures and their implementation in physics and biology, have been discussed in several contributions to these Proceedings.

In this article, we shall persue a geometrical perspective [12,13,18] which facing the main problem of the study of dissipative systems, stems historically from Einstein's theory of Brownian motion [9].

While to Brown fluctuations seemed to be the inprint of biological systems, the development of the theoretical studies of fluctuations goes back to Batchelier investigations of the Paris stock market. Posteriorly, in this century, fluctuations came to be the subject of investigation by physicists (Fokker, Planck, Langevin, etc.) and mathematicians alike (Kolmogorov, Feller, Ito, etc.). Yet, it is from the theory of linear non-equilibrium thermodynamics developed by Onsager in the thirties, and by Prigogine in the forties, that it is today acknowledged that fluctuations have a universality that encompasses as much as chemical reactions, electrical circuits, population biology, control systems, etc. [1,2,25], not to mention the most basic role they have in quantum gravity [15] and the renormalization of gauge theories.

This article will deal with the fact that non-linear fluctuations of Markovian type

can be seen equivalently as geometrical space-time structures. More specifically, this article will deal with the problem of the large scale geometrical structure of the Universe and its relation with the local small-scale structure of dissipative systems. The problem of irreversibility and the existance of a local arrow of time, will be approached from this perspective.

We shall show that the introduction of a Riemann-Cartan-Weyl connection of the most general type, with metric and a torsion tensor reduced to its more general trace-torsion term, accounts for the formulation of irreversibility, and furthermore, to the non-linear fluctuation-dissipation equations of statistical mechanics. We shall furthermore, relate the growth of entropy, to the trace-torsion, in a formulation of a non-linear H-theorem, which by-passes completely the reversibility assumptions on phase-space originally assumed by Boltzmann. In all respects, this article will present a complete formulation of the central problems of statistical mechanics, stemming from the geometrical structures just described. Thus our theory proposes a unified treatment of stochasticity and determinism.

We shall present a self-contained presentation of the probabilistic structures involved. Thus, sections I and II, introduce the preliminary concepts and serve as the motivation for the posterior geometrical perspective of this article.

I. PRELIMINARIES

Suppose we have random variables $\xi_1, \ldots, \xi_r$, (r is an arbitrary positive integer) with joint probability density $p(\xi_1, \ldots, \xi_r)$. The mean value

$$< \xi_1 \ldots \xi_r >= \int \xi_{\alpha_1} \ldots \xi_{\alpha_r} p(\xi_1, \ldots, \xi_r) d\xi_1 \ldots d\xi_r$$

is called the **moment**.

The **characteristic function** is defined by

$$\Xi(iu_1, \ldots, iu_n) =< exp(\sum_{\alpha=1}^{n} iu_\alpha \xi_\alpha) > \tag{1.1}$$

$$= \int exp(\sum_{\alpha=1}^{n} iu_\alpha \xi_\alpha) p(\xi_1, \ldots, \xi_n) d\xi_1 \ldots d\xi_r. \tag{1.2}$$

Then,

$$< \xi_{\alpha_1} \ldots \xi_{\alpha_n} >= i^{-n} [\frac{\partial \Xi(iu_1, \ldots, iu_n)}{\partial u_{\alpha_1} \ldots \partial u_{\alpha_n}}]|_{u=0}, 1 \leq n \tag{1.3}$$

where, $u = 0$ means $u_1 = \ldots = u_n = 0$.

If the characteristic function is analytic at $u \equiv 0$, then, the Taylor formula is valid

$$\Xi(iu_1, \ldots, iu_r) = 1 + \sum_{m=1}^{\infty} \frac{i^m}{m!} \sum_{\alpha_1 \ldots \alpha_m} < \xi_{\alpha_1} \ldots \xi_{\alpha_r} > u_{\alpha_1} \ldots u_{\alpha_m}. \tag{1.3}$$

Thus, the characteristic function can be expressed in terms of the moments.

The **correlator**

$$< \xi_{\alpha_1}, \ldots, \xi_{\alpha_r} > := i^{-m} [\frac{\partial^r ln\ \Xi(iu)}{\partial u_{\alpha_1} \ldots \partial u_{\alpha_r}}]|_{u \equiv 0} \tag{1.4}$$

We note that the correlator of the random variables $\xi_{\alpha_1}, \ldots \xi_{\alpha_m}$ is 0 if these variables can be divided into two groups such that one group is statistically independent from the other group. If, in fact, $\xi_{\alpha_1} \ldots \xi_{\alpha_k}$ are statistically independent of $\xi_{\alpha_{k+1}, \ldots, \xi_{\alpha_m}}$, the probability density is

$$p(\xi_{\alpha_1}, \ldots, \xi_{\alpha_m}) = p_1(\xi_{\alpha_1}, \ldots, \xi_{\alpha_k}).p_2(\xi_{\alpha_{k+1}}, \ldots, \xi_{\alpha_m}) \tag{1.5}$$

and the characteristic function factorizes as

$$\Xi(iu_{\alpha_1}, \ldots, iu_{\alpha_m}) = \Xi_1(iu_{\alpha_1}, \ldots, iu_{\alpha_k}) \Xi_2(iu_{\alpha_{k+1}}, \ldots, iu_{\alpha_m}).$$

Then, writing $v_\alpha = iu_\alpha$, for any α, we have

$$ln\Xi(v_{\alpha_1}, \ldots, v_{\alpha_m}) = ln\Xi(v_{\alpha_1}, \ldots, v_{\alpha_k}) + ln\Xi_2(v_{\alpha_{k+1}}, \ldots, v_{\alpha_m})$$

and then

$$< \xi_{\alpha_1}, \ldots, \xi_{\alpha_m} >= \frac{\partial^m ln\Xi_1(v_{\alpha_1}, \ldots, v_{\alpha_n})}{\partial v_{\alpha_1} \ldots \partial v_{\alpha_k} \partial v_{\alpha_{k+1}} \ldots \partial_{\alpha_m}} + \frac{\partial ln\Xi_2(v_{\alpha_{k+1}}, \ldots, v_m)}{\partial \alpha_1 \ldots \partial v_{\alpha_k} \partial v_{\alpha_{k+1}} \ldots \partial v_{\alpha_m}} \equiv 0$$

We have from (1.3, 1.4) (if the expression converges)

$$\Xi(v_1, \ldots, v_n) = exp(1 + \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{\alpha_1, \ldots \alpha_n = 1}^{n} < \xi_{\alpha_1}, \ldots, \xi_{\alpha_m} > v_{\alpha_1} \ldots v_{\alpha_m}) \tag{1.6}$$

Therefore,

$$1 + \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{\alpha_1, \ldots, \alpha_m = 1}^{n} < \xi_{\alpha_1} \ldots \xi_{\alpha_m} > v_{\alpha_1} \ldots v_{\alpha_m}$$

$$= exp(\sum_{k=1}^{m} \frac{1}{k!} \sum_{\beta_1, \ldots \beta_k = 1}^{n} < \xi_{\beta_1}, \ldots, \xi_{\beta_k} > v_{\beta_1}, \ldots, v_{\beta_k}). \tag{1.6'}$$

This is the generating equation, since multiple partial differentiation with respect to the components of $v = (v_1, \ldots, v_n)$, and setting $v \equiv 0$, we can express the moments in terms of the correlators. Thus, double differentiation yields

$$< \xi_{\alpha_1}\xi_{\alpha_2} >=< \xi_{\alpha_1}, \xi_{\alpha_2} > + < \xi_{\alpha_1} >< \xi_{\alpha_2} >, \tag{1.8}$$

and so on.

II. MARKOV PROCESSES

Let $x(\tau) = (x_1(\tau), \ldots, x_r(\tau))$ be an r-component process. It is described by the set of many-time probability densities

$$p(x(\tau), p(x(\tau_1), x(\tau_2)), \ldots, p(x(\tau_1), \ldots, x(\tau_n)), \ldots. \tag{2.1}$$

Next, we introduce the conditional probability density $p(\eta|\xi)$, i.e. the probability density of the random variable η at the fixed value of ξ, defined by

$$p(\eta|\xi) = \frac{p(\eta, \xi)}{p(\xi)} \tag{2.3}$$

Thus, putting $\eta = x(\tau_2), \xi = x(\tau_1)$, we have

$$p(x(\tau_1), x(\tau_2)) = p(x(\tau_2)|x(\tau_1))p(x(\tau_1)). \tag{2.4}$$

If instead we put $\xi = x(\tau_3), \eta = (x(\tau_2), x(\tau_1))$, we have from (2.1)

$$p(x(\tau_1), x(\tau_2), x(\tau_3)) = p(x(\tau_3)|x(\tau_1), x(\tau_2)).p(x(\tau_2)|x(\tau_1)).p(x(\tau_1)). \tag{2.5}$$

Generally, we have

$$\begin{aligned} p(x(\tau_1), \ldots, x(\tau_n)) = p(x(\tau_n)|x(\tau_1), \ldots, x(\tau_{n-1})) \ldots p(x(\tau_3)|x(\tau_1), x(\tau_2)) \\ .p(x(\tau_2)|x(\tau_1))p(x(\tau_1)). \end{aligned} \tag{2.6}$$

Assume $\tau_1 < \tau_2 < \ldots < \tau_n$. A r-component stochastic process $x(\tau)$ is called **Markovian**, if for any $k \in R$,

$$p(x(\tau_k)|x(\tau_1), \ldots, x(\tau_{k-1})) = p(x(\tau_k)|x(\tau_{k-1})), \tag{2.7}$$

is valid. This condition is usually framed in saying that Markovian processes are memoryless. In this case,

$$\begin{aligned} p(x(\tau_1), \ldots, x(\tau_n)) = p(x(\tau_n)|x(\tau_{n-1})).p(x(\tau_{n-1}|x(\tau_{n-2})) \ldots p(x(\tau_3)|x(\tau_2)) \\ p(x(\tau_2)|x(\tau_1)p(x(\tau_1)) \end{aligned} \tag{2.8}$$

The, we conclude that for a Markovian process, the many-time probability densities, are determined by the one-time probability density $p(x(\tau))$ and the two-time conditional probability density $p(x(\tau)|x(\tau')), \tau' \leq \tau$, i.e. the transition probability density (in quantum mechanics it is called the "heat kernel").

Now, if we integrate the three-time $p(x(\tau_1), x(\tau_2), x(\tau_3))$ with respect to $x(\tau_2)$ on R, to obtain the two-time probability density

$$\int p(x(\tau_1), x(\tau_2), x(\tau_3))dx(\tau_2) = p(x(\tau_1), x(\tau_3)). \tag{2.9}$$

Factorizing both densities as in (2.8), we obtain the equation ($\tau_1 < \tau_2 < \tau_3$)

$$\int p(x(\tau_1)|x(\tau_2))p(x(\tau_2)|x(\tau_1))dx(\tau_2) = p(x(\tau_3)|x(\tau_1)) \tag{2.10}$$

This is the Chapman-Kolmogorov-Smoluchowski (CKS) equation which is satisfied by the transition probabilities

Denoting $x(\tau_1) = x", x(\tau_2) = x'$, we can write the transition probability density $p(x(\tau_2)|x(\tau_1))$ as $p_{\tau_2\tau_1}(x', x")$. Then, the CKS equation yields for $x = x(\tau_3), x' = x(\tau_2), x" = x(\tau_1)$, the equation

$$\int p_{\tau_3\tau_2}(x|x')p_{\tau_2\tau_1}(x'|x")dx' = p_{\tau_3\tau_1}(x|x"). \tag{2.11}$$

Let us introduce the conditional characteristic function of increments $\triangle x = x - x' = x(\tau_3) - x(\tau_2)$, as the Fourier transform of $p_{\tau_3\tau_2}(x|x')$:

$$\Xi_{\tau_3\tau_2}(iu|x') = \int exp(iu\triangle y)p_{\tau_3\tau_2}(x|x')dx, \tag{2.12}$$

where $u\triangle x = \sum_{\alpha=1}^{r} u_\alpha \triangle x_\alpha$. Using the inverse transformation one can express the transition probability density $p_{\tau_3\tau_2}(x|x')$ in terms of the characteristic function

$$p_{\tau_3\tau_2}(x|x') = (2\pi)^{-r} \int exp[-iu(x - x')]\Xi_{\tau_3\tau_2}(iu|x")du \tag{2.13}$$

Inserting this into the CKS equation we get

$$(2\pi)^{-r} \int \Xi_{\tau_3\tau_2}(iu|x")exp[-iu(x - x')]p_{\tau_2\tau_1}dudx' = p_{\tau_3\tau_1}(x|x") \tag{2.14}$$

This last equation is easily seen to be equal to

$$(2\pi)^{-r} \int \Xi_{\tau_3\tau_2}(-\frac{\partial}{\partial x'}|x')exp[-iu(x - x')]p_{\tau_2\tau_1}(x'|x")dudx' = p_{\tau_3\tau_1}(x|x"). \tag{2.15}$$

We integrate this last equation with respect to u by applying the integral representation of the delta function:

$$(2\pi)^{-r}\int_{-\infty}^{\infty} exp(iuz)du = \delta(z) \equiv \delta(z_1)\dots\delta(z_r); \tag{2.16}$$

Hence, we have

$$\int(-\frac{\partial}{\partial x}|x')\delta(x-x')p_{\tau_2\tau_1}(x'|x")dx' = p_{\tau_3\tau_1}(x|x"). \tag{2.17}$$

This integration trivially yields the CKS characteristic function equation

$$N_{\partial,x}\Xi_{\tau_3\tau_2}(-\frac{\partial}{\partial x}|x)p_{\tau_3\tau_1}(x|x") = p_{\tau_3\tau_1}(x|x"). \tag{2.18}$$

The last inserted symbol, $N_{\partial,x}$ indicates that the differentiation is done after multiplication by x of the transition density has been carried out. Using now the Taylor expansion (1.6′) of the characteristic function, we obtain the following form of the CKS equation

$$\begin{aligned} p_{\tau_3\tau_1}(x|x") = p_{\tau_2\tau_1}(x|x") + \sum_{m=1}^{\infty}\frac{(-1)^{-m}}{m!}\sum_{\alpha_1\dots\alpha_m=1}^{r} \\ \frac{\partial^m}{\partial x^{\alpha_1}\dots\partial x^{\alpha_m}}[<\triangle x_{\alpha_1}\dots\triangle x_{\alpha_m}>_x p_{\tau_2\tau_1}(x|x")] \end{aligned} \tag{2.19}$$

Here the subscript x shows that the moments are in fact conditional, i.e. they are taken at fixed values of $x = x(\tau_2)$.

Let $\tau = \tau_3 - \tau_2 > 0$. We write the CKS characteristic function equation in the form

$$\tau^{-1}[p_{\tau_3\tau_1}(x,x") - p_{\tau_2\tau_1}(x|x")] = \tau^{-1}\left(N_{\partial,x}\Xi_{\tau_3\tau_2}(-\frac{\partial}{\partial x}|x) - 1\right)p_{\tau_2\tau_1}(x|x"). \tag{2.19}$$

If we let τ tend to zero in this equation, provided the limit

$$\Phi_{\tau_2}(v,x) = lim_{\tau\to 0}\frac{\Xi_{\tau_2+\tau}(v|x) - 1}{\tau} \tag{2.20}$$

exists, we obtain the equation

$$\frac{\partial}{\partial\tau_2}p_{\tau_2\tau_1} = N_{\partial,x}\Phi_{\tau_2}(-\frac{\partial}{\partial x},x)p_{\tau_2\tau_1}(x|x"). \tag{2.21}$$

This is called the **master equation**. The initial condition for the master equation is

$$p_{\tau_2\tau_1}(x|x") = \delta(x-x"), \text{for } \tau_2 = \tau_1. \tag{2.22}$$

Therefore, if the function $\Phi_\tau(v,x)$ is known, one can find the transition densities as the fundamental solution of the master equation. We shall call this function as the **Markov generator function**.

Thus, we see that the operator $L_\tau = N_{\partial,x}\Phi_\tau(-\frac{\partial}{\partial x},x)$ defines the statistics of the Markov process. This operator, is called the Markov operator, or still the **differential operator** or the **infinitesimal generator of the Markov process**.

Thus, it is the fundamental structure for the construction of the Markov processes. Yet, we remark that the present derivation has the most serious problem of being non-covariant.

Multiplying the master equation by $p_{\tau_1}(x") = p(x(\tau_1))$, and further integrating with respect to $x" = x(\tau_1)$, we find that the one-time probability density

$$p_\tau(x) = \int p_{\tau\tau_1}(x|x")p_{\tau_1}(x")dx", \tag{2.23}$$

satisfies the analogous master equation

$$\frac{\partial p_\tau(x)}{\partial \tau} = N_{\partial,x}\Phi_\tau(-\frac{\partial}{\partial x},x)p_\tau(x). \tag{2.24}$$

The previous process of taking the limit $\tau \to 0$ can also be carried out for $p_\tau(x)$. Then, the master equation (2.19) will take the form

$$\frac{\partial p_\tau(x)}{\partial \tau} = \sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\sum_{\alpha_1\ldots\alpha_m=1}^{r}\frac{\partial^m}{\partial x_{\alpha_1}\ldots\partial x_{\alpha_m}}[\Lambda^{\alpha_1,\ldots,\alpha_m}p_\tau(x)], \tag{2.25}$$

where

$$\Lambda^{\alpha_1\ldots\alpha_m}(x) = lim_{\tau\to 0} < \triangle x_{\alpha_1}\ldots\triangle x_{\alpha_m} >_x . \tag{2.26}$$

The Markov generator function $\Phi(v,x)$ and the coefficients $\Lambda^{\cdots}$ are related by

$$\Phi(v,x) = \sum_{m=1}^{\infty}\frac{1}{m!}\sum_{\alpha_1,\ldots,\alpha_m}\Lambda^{\alpha_1\ldots\alpha_m}(x)v_{\alpha_1}\ldots v_{\alpha_m}, \tag{2.27}$$

if the series converges. In this case this identity is called the Kramers-Moyal equation. The functions $\Lambda^{\cdots}$ determine the statistics of the process, and are called the **coefficients of the master equation**, or still, the **coefficient functions**.

The master equation is called the Fokker-Planck equation when the only non-zero coefficients are $\Lambda^\alpha(x)$ and $\Lambda^{\alpha\beta}(x)$. Thus, the Fokker-Planck equation is

$$\frac{\partial p_\tau(x)}{\partial \tau} = -\frac{\partial}{\partial x^\alpha}[\Lambda^\alpha(x)p_\tau(x)] + \frac{1}{2}\frac{\partial^2}{\partial x^\alpha\partial x^\beta}[\Lambda^{\alpha\beta}(x)p_\tau(x)]. \tag{2.28}$$

Here we have used the Einstein convention of repeated indices summation.

A random process $x = (x^1(\tau), \ldots, x^r(\tau))$ is called **stationary** if all many-time probability densities $p(x(\tau_1), \ldots, x(\tau_n))$ are constant in time, i.e. if the substitution $\tau_1 + a_1, \ldots, \tau_n + a_n$ for $\tau_1, \ldots, \tau_n$ leaves the probability densities unchanged. In this case, the many-time probability densities depend only on $(n-1)$ time differences, say, in $\tau_2 - \tau_1 = \tau_{21}, \ldots, \tau_n - \tau_1 = \tau_{n1}$. Considering (2.8) we see that the stationarity condition for the Markov processes reduces to the following two conditions:

i)The one-time probability density $p(x(\tau))$ is actually independent of τ.

ii) The transition probabilities $p_{\tau\tau'}$ depends only in the time differences $s = \tau - \tau'$.

Then, the characteristic function for the stationary case must depend only on the time differences $\tau_3 - \tau_2$, and the Markov generator function Φ_τ in (2.20) is actually independent of τ. In the stationary case, the master equation (2.24) is of the form

$$\frac{\partial p_\tau(x|x')}{\partial \tau} = N_{\partial,x}\Phi(-\frac{\partial}{\partial x}, x)p_\tau(x|x'), \tag{2.29}$$

which supplemented by the initial condition

$$p_\tau(x|x") = \delta(x - x") \text{ for } \tau = 0, \tag{2.30}$$

determines completely the transition probabilities $p_\tau(x|x")$. Then, the equation (2.21) written for the one-time stationary (also called **equilibrium**) probability density, $p_{eq}(x)$ has the form

$$N_{\partial,x}\Phi(-\frac{\partial}{\partial x}, x)p_{eq}(x) = 0, \tag{2.31}$$

since $p_{eq}(x)$ is constant in time. This equation and the normalization condition

$$\int p_{eq}(x)dx = 1, \tag{2.32}$$

determines completely the stationary one-time probability density. Thus we see that the operator of the master equation, i.e. the Markov generator function $\Phi(v, x)$ completely determines the statistical properties of the stationary Markov process.

Note that the Fokker-Planck equation involves the natural adjoint (with respect to the standard Hilbert space pairing) of a second-order differential operator with null zero-order term. It is this operator we want to set out in covariant form, and this will be the subject of the next chapter, with the subsequent introduction of the Lie-isotopic geometry to give account of the covariant Fokker-Planck operator.

III.A. THE INVARIANT FOKKER-PLANCK EQUATION AND RIEMANN-CARTAN-WEYL GEOMETRIES

Let us consider for a start, a smooth n-dimensional compact orientable manifold M, on which we shall consider a second-order smooth differential operator L. On a local coordinate system, $(x^\alpha), \alpha = 1, \ldots, n$, L is written as

$$L = \frac{1}{2} g^{\alpha\beta}(x)\partial_\alpha\partial_\beta + B^\alpha(x)\partial_\alpha + c(x). \qquad (3.1).$$

From now on, we shall fix this coordinate system, and all local expressions shall be written in it.

Although formally, there is no restriction as to the nature of M, we are really thinking on a four-dimensional space-time manifold, or still, a three-dimensional space manifold and **not** in a phase-space manifold. Our presentation expands on [12,13].

The *principal symbol* σ of L, is the section of the bundle of real bilinear symmetric maps on T^*M, defined as follows: for $x \in M$, $p_i \in T_x^*M$, take C^2 functions, $f_i : M \to R$ with $f_i(x) = 0$ and $df_i(x) = p_i$, $i = 1, 2$; then,

$$\sigma(x)(p_1, p_2) = L(f_1 f_2)(x).$$

Note that σ is well defined, i.e., it is independent of the choice of the functions $f_i, i = 1, 2$.

If L is locally as in (1), then σ is locally represented by the matrix $(g^{\alpha\beta})$. We can also view σ as a section of the bundle of linear maps $L(T^*M, TM)$, or as a section of the bundle $TM \otimes TM$, or still as a bundle morphism from T^*M to TM. If σ is a bundle isomorphism, it induces a pseudo-Riemannian structure g on M, $g : M \to L(TM, TM)$:

$$g(x)(v_1, v_2) := < \sigma(x)^{-1} v_1, v_2 >_x,$$

for $x \in M$, $v_1, v_2 \in T_x^*M$. Here, $< ., . >_x$ denotes the natural duality between T_x^*M and T_xM. Locally, $g(x)$ is represented by the matrix $\frac{1}{2}(g^{\alpha\beta}(x))^{-1}$. Consider the quadratic forms over M associated to L, defined as

$$Q_x(p_x) = \frac{1}{2} < p_x, \sigma_x(p_x) >_x,$$

for $x \in M$, $p_x \in T_x^*M$. Then, with the local representation (3.1) for L, Q_x is represented as $\frac{1}{2}(g^{\alpha\beta}(x))$. Then, L is an elliptic (semi-elliptic) operator whenever for all $x \in M$, Q_x is positive-definite (non-negative definite). We shall assume in the following that L is an elliptic operator. In this case, σ is a bundle isomorphism and the metric g is actually a Riemannian metric. Notice, as well, that $\sigma(df) = grad\, f$, for any $f : M \to R$ of class C^2, where grad denotes the Riemannian gradient.

We wish to give an intrinsic description of L, i.e. a description independent of the local coordinate system. This is the essential prerequisite of covariance.

For this, we shall introduce an arbitrary connection on M, whose covariant derivative we shall denote as ∇. We remark here that ∇ need **not** be the Levi-Civita connection associated to g; we shall precise this below. Let $\sigma(\nabla)$ denote the second-order part of L, and let us denote by $X_0(\nabla)$ the vector field on M given by the first-order part of L. Finally, the zero-th order part of L is given by $L(1)$, where 1 denotes the constant function on M equal to 1.

Then, for $f : M \to R$ of class C^2, we have

$$\sigma(\nabla)(x) = \frac{1}{2} trace(\nabla^2 f)(x) = \frac{1}{2}(\nabla df)(x)), \tag{3.2}$$

where the trace is taken in terms of g, and ∇df is thought as a section of $L(T^*M, T^*M)$. Also, $X_0(\nabla) = L - L(1) - \sigma(\nabla)$. If $\Gamma^\alpha_{\beta\gamma}$ is the local representation for the Christoffel symbols of the connection, then the local representation of $\sigma(\nabla)$ is:

$$\sigma(\nabla)(x) = \frac{1}{2} g^{\alpha\beta}(x)(\partial_\alpha \partial_\beta + \Gamma^\gamma_{\alpha\beta}(x)\partial_\gamma), \tag{3.3}$$

and,

$$X_0(\nabla)(x) = B^\alpha(x)\partial_\alpha - \frac{1}{2} g^{\alpha\beta}(x)\Gamma^\gamma_{\alpha\beta}\partial_\gamma. \tag{3.4}$$

If ∇ is the Levi-Civita connection associated to g, which we shall denote as ∇^g, then for any $f : M \to R$ of class C^2:

$$\sigma(\nabla^g)(df) = \frac{1}{2} tr((\nabla^g)^2 f) = \frac{1}{2} tr(\nabla^g df) = -1/2 \mathrm{div_g\ grad}\ f = 1/2 \triangle_g f. \tag{3.5}$$

Here, $\triangle_g$ is the Levi-Civita laplacian operator on functions; locally,it is written as

$$\triangle_g = g^{-1/2}\partial_\alpha((g^{1/2} g^{\alpha\beta}\partial_\beta); \quad g = det(g_{\alpha\beta}), \tag{3.6}$$

and div_g is the Riemannian divergence operator on vector fields $X = X^\alpha(x)\partial_\alpha$:

$$div_g(X) = -g^{-1/2}\partial_\alpha(g^{1/2} X^\alpha). \tag{3.7}$$

Note the important relation which we already have used in eqt. (2.5) and will be used repeatedly; namely:

$$div_g(X) = -\delta \tilde{X},$$

where δ is the co-differential operator (see (3.19) below), and $\tilde{X}$ is the one-form conjugate to the vector field X, i.e. $\tilde{X}_\alpha = g_{\alpha\beta} X^\beta$.

We now take ∇ to be a Riemann-Cartan connection with torsion [11], which we additionally assume to be compatible with g, i.e. $\nabla g = 0$. Then $\sigma(\nabla) = \frac{1}{2} tr(\nabla^2)$. Let us compute this. Denote the Christoffel coefficients of ∇ as $\Gamma^{\alpha}_{\beta\gamma}$; then,

$$\Gamma^{\alpha}_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} + 1/2 K^{\alpha}_{\beta\gamma}, \tag{3.8}$$

where the first term in (3.8) stands for the Christoffel Levi-Civita coefficients of the metric g, and

$$K^{\alpha}_{\beta\gamma} = T^{\alpha}_{\beta\gamma} + S^{\alpha}_{\beta\gamma} + S^{\alpha}_{\gamma\beta}, \tag{3.9}$$

is the cotorsion tensor, with $S^{\alpha}_{\beta\gamma} = g^{\alpha\nu} g_{\beta\kappa} T^{\kappa}_{\nu\gamma}$, and $T^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\gamma\beta}$ the skew-symmetric torsion tensor.

Let us consider the Laplacian operator associated to this Cartan connection, defined -in extending the usual definition- by

$$H(\nabla) = 1/2 g^{\alpha\beta} \nabla_{\alpha} \nabla_{\beta}, \tag{3.10}$$

where ∇ stands for the covariant derivative operator with respect to Γ; then, $\sigma(\nabla) = H(\Gamma)$. A straightforward computation shows that that $H(\Gamma)$ only depends in the trace of the torsion tensor and g:

$$H(\Gamma) = 1/2 \triangle_g + g^{\alpha\beta} Q_{\beta} \partial_{\alpha}, \tag{3.11}$$

with $Q = T^{\nu}_{\nu\beta} dx^{\beta}$, the trace-torsion one-form.

Therefore, for the Riemann-Cartan connection ∇ defined in (3.8), we have that

$$\sigma(\nabla) = \frac{1}{2} tr(\nabla^2) = \frac{1}{2} \triangle_g + \hat{Q}, \tag{10}$$

with $\hat{Q}$ the vector-field conjugate to the 1-form Q: $\hat{Q}(f) = < Q, \mathrm{grad}\, f >$, $f : M \to R$. In local coordinates,

$$\hat{Q}^{\alpha} = g^{\alpha\beta} Q_{\beta}.$$

We further have:

$$X_0(\nabla) = B - \frac{1}{2} g^{\alpha\beta} \left\{ \begin{matrix} \gamma \\ \alpha\beta \end{matrix} \right\} \partial_{\gamma} - \hat{Q}, \tag{3.12}$$

Therefore, the invariant decomposition of L is

$$\frac{1}{2} tr(\nabla^2) + X_0(\nabla) + L(1) = \frac{1}{2} \triangle_g + b + L(1). \tag{3.13}$$

with

$$b = B - \frac{1}{2} g^{\alpha\beta} \left\{ \begin{matrix} \gamma \\ \alpha\beta \end{matrix} \right\} \partial_\gamma .$$

Notice that (3.12) can be thought as arising from a gauge transform... with $\tilde{b}$ the 1-form conjugate to b.

If we take for a start ∇ with Christoffel symbols of the form

$$\Gamma^\alpha_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} + \frac{2}{(n-1)} \{ \delta^\alpha_\beta \, Q_\gamma - g_{\beta\gamma} \, Q^\alpha \} \tag{3.15}$$

with

$$Q = \tilde{b}, \quad \text{i.e.} \quad \hat{Q} = b, \tag{3.16}$$

we have

$$X_0(\nabla) = 0$$

and

$$\sigma(\nabla) = \frac{1}{2} tr(\nabla^2) = H(\nabla) = \frac{1}{2} tr((\nabla^g)^2) + \hat{Q} = \frac{1}{2} \triangle_g + b. \tag{3.16'}$$

Therefore,

$$L = \sigma(\nabla) + L(1) = \frac{1}{2} tr((\nabla^g)^2) + \hat{Q} + L(1). \tag{3.17}$$

The restriction we have placed in ∇ to be as in (3.15), i.e. only the trace component of the irreducible decomposition of the torsion tensor is taken, is due to the fact that all other components of this tensor do not appear at all in the laplacian of (the otherwise too general) ∇. In the particular case of dimension 2, this is automatically satisfied. In the case we actually have assumed that g is Riemannian, the expression (3.17) is the most general invariant laplacian acting on functions defined on a smooth manifold. This restriction, will allow us to establish a one-to-one correspondance between Riemann-Cartan connections of the form (3.15) with Markovian diffusion processes. These metric compatible connections we shall call RCW geometries (short for Riemann-Cartan-Weyl), since the trace-torsion is a Weyl 1-form [12]. Thus, these geometries do not have the historicity problem which lead to Einstein's rejection of the first gauge theory ever proposed by Weyl.

To obtain the most general form of the RCW laplacian, we only need to know the most general decomposition of one-forms on a smooth compact manifold. The answer to this problem, is given by the famous de Rham-Kodaira-Hodge theorem [22], which we present now.

Let us assume in the following that we have a smooth n-dimensional orientable compact manifold M provided with a Riemannian metric, g. We consider the Hilbert

space of square summable ω of smooth differential forms of degree q on M, with respect to vol_g. We shall denote this space as $L^{2,q}$ or still as $L^2\Omega^q(M, \mathrm{vol}_g)$. The inner product is

$$< \omega, \phi >= \int_M < \omega(x), \phi(x) > \mathrm{vol}_g, \tag{3.18}$$

where the integrand is given by the natural pairing between the components of ω and the conjugate tensor: $g^{\alpha_1\beta_1} \ldots g^{\alpha_q\beta_q}\phi_{\beta_1\ldots\beta_q}$; alternatively, we can write in a coordinate independent way: $< \omega(x), \phi(x) > vol_g = \omega(x) \wedge *\phi(x)$, with $*$ the Hodge star operator, for any $\omega, \phi \in L^{2,q}$ [22].

The de Rham-Kodaira operator on $L^{2,q}$ is defined as

$$\triangle = -(d + \delta)^2 = -(d\delta + \delta d), \tag{3.19}$$

where δ is the formal adjoint defined on $L^{2,q+1}$ of the exterior differential operator d defined on $L^{2,q}$:

$$< \delta\phi, \omega >=< \phi, d\omega >,$$

for $\phi \in L^{2,q+1}$ and $\omega \in L^{2,q}$. Then,

$$\delta^2 = 0.$$

In the case of $q = 0$, the de Rham-Kodaira operator coincides with the Laplace-Beltrami operator on functions encountered before; in the general case we have in addition of $tr(\nabla^g)^2$ the contribution of the Weitzenbock curvature term we shall describe below.

The de Rham-Kodaira-Hodge theorem states that $L^{2,1}$ admits the following invariant decomposition. Let $\omega \in L^{2,1}$; then,

$$\omega = d\, f + A_1 + A_2, \tag{3.20}$$

where $f : M \to R$ is a smooth function on M, A_1 is a co-closed smooth 1-form:

$$\delta A_1 = -div_g \hat{A}_1 = 0,$$

and A_2 is a co-closed and closed smooth 1-form:

$$\delta A_2 = 0, dA_2 = 0. \tag{3.21}$$

Otherwise stated, A_2 is an harmonic one-form, i.e.

$$\triangle A_2 = 0. \tag{3.22}$$

Furthermore, this decomposition is orthogonal in $L^{2,1}$, i.e.:

$$< d\, f,\ A_1 >=< d\, ln\psi,\ A_2 >=< A_1, A_2 >= 0. \tag{3.23}$$

Now we return to the problem of characterizing the most general form of the trace torsion one-form Q. By the de Rham-Kodaira-Hodge theorem, there exists a positive smooth function ψ defined on M, and one-forms A_1, A_2 on M, which are co-closed and harmonic, respectively, and such that $d\, ln\, \psi, A_1$ and A_2 are orthogonal in $L^{2,1}$, so that the trace-torsion of the connection ∇ is

$$Q = d\, ln\, \psi + A_1 + A_2 \tag{3.24}$$

We have taken the function f on the decomposition to be of a real phase of the form $ln\ \psi$, for reasons that we shall see justify below, when determining the forms of A_1 and A_2. To avoid notational overloading, we prefer to explicit already this choice.

We observe that this choice of the exact component of the trace-torsion, corresponds to the torsion associated to the Lie-isotopic orthogonal group unit,

$$\hat{I} = \psi^2 I,$$

appearing in the Gasperini-Santilli Lie-isotopic gauge theory [7,8].

We shall denote from now on the Laplacian operator $H(\nabla)$ associated to the Riemann-Cartan-Weyl connection (3.15), with Q given by (3.24), as $H(g, \psi, A_1, A_2)$.

We are interested in Markovian semigroups [20] $\{P_\tau, \tau \geq 0\}$ with infinitesimal generator given by $H(\Gamma)$ of (3.16′):

$$H(\Gamma)f = \text{str } \lim_{\tau \to 0} \frac{P_\tau f - f}{\tau}$$

for f in the domain of $H(\nabla)$; here, the limit is taken in the strong (operator) sense. We further assume that $\{P_\tau, \tau \geq 0\}$ preserves probability, i.e. $P_\tau(1) = 1$, for any $\tau \geq 0$; consequently, $H(\Gamma)(1) = 0$, i.e. the zero-order ("potential") term of the operator $H(\Gamma)$ is identically zero. This is the situation of the Fokker-Planck equation.

The role of the trace-torsion vector field

$$b = \hat{Q} = grad\ ln\psi + \hat{A}_1 + \hat{A}_2, \tag{3.25}$$

the vector field conjugate to the trace-torsion 1-form, is that of the drift (average velocity) of the continuous sample curves of the diffusion processes associated to $H(g, \psi, A_1, A_2)$. Therefore, the introduction of the torsion is a most essential feature of the diffusion

processes associated to $\{P_\tau, \tau \geq 0\}$, since Brownian processes have continuous non-differentiable sample paths. (Actually, they are fractals).

We must remark that τ is not to be confused with the relativistic time coordinate of M; it is to be thought as an internal time evolution parameter of the diffusion we shall describe below, consistently with our introduction of the master equation; this distinction between τ and the relativistic time coordinate was originally conceived in quantum gravity by B. de Witt [15]. This time parameter is Liouville's time in Prigogine's theory of non-equilibrium statistical mechanics [4,19].

Therefore, the most general invariant laplacian $H(\nabla)$ on a compact smooth manifold depends on g, ψ, A_1 and A_2; thus we write

$$H(\nabla) = H(g, \psi, A_1, A_2);$$

then, acting on functions ϕ defined on M,

$$H(g,\psi,A_1,A_2)\phi = 1/2\triangle_g\phi + < grad\, ln\psi,\ grad\,\phi > + < A_1,\ grad\,\phi > + < A_2,\ grad\,\phi >, \tag{3.26}$$

or still, in local coordinates,

$$H(g,\psi,A_1,A_2)\phi = 1/2 g^{\alpha\beta}(\partial_\beta ln\ \psi + [A_{1\ \beta} + A_{2\ \beta}])\partial_\alpha\phi,$$

where $A_i = A_{i\ \alpha} dx^\alpha$, $i = 1, 2$.

Then, the transpose of $H(g, \psi, A_1, A_2)$ in $L^{2,0}$ is the operator acting on smooth functions ϕ on M:

$$H(g,\psi,A_1,A_2)^\dagger\phi = \frac{1}{2}\triangle_g\phi - div_g(\phi.grad\ ln\ \psi) - div_g(\phi.[\hat{A}_1 + \hat{A}_2]), \tag{3.27}$$

where we have kept the above notation, so that $\hat{A}$ is the vector field conjugate to A.

The transition density $p_\tau^\nabla(x, y)$ is determined as the fundamental solution of the "heat" equation on the first variable x:

$$\frac{\partial u}{\partial \tau} = H(g,\psi,A_1,A_2)(x)u\ . \tag{3.28}$$

It will be very important for the following, to note that the semigroup $\{P_\tau : \tau \geq 0\}$ has a unique τ- independent **invariant** probability density ρ determined as the fundamental weak solution (in the sense of the theory of generalized functions) of the τ-independent Fokker-Planck equation:

$$H(g,\psi,A_1,A_2)^\dagger(\rho) = 0; \tag{3.29}$$

cf. (2.28). Let us determine ρ. We choose a smooth function U defined on M such that

$$H(g,\psi,A_1,A_2)^\dagger(e^{-U}) = 0,$$

i.e. $\rho = e^{-U}vol_g$ is an invariant measure. Since,

$$H(g,\psi,A_1,A_2)^\dagger = -1/2\delta d(e^{-U}) + \delta(e^{-U}\hat{Q}) = \frac{-1}{2}\delta d(e^{-U}) + \delta(e^{-U}(grad\ ln\ \psi + \hat{A}_1 + \hat{A}_2)) = 0,$$

i.e.

$$-1/2\delta(\ d(e^{-U}) + e^{-U}[\ grad\ ln\ \psi + \hat{A}_1 + \hat{A}_2]) = 0,$$

and finally, we have

$$e^{-U}(d\ ln\ \psi + A_1 + A_2) - \frac{1}{2}d(e^{-U}) = \delta\beta_2 + \omega_1, \tag{3.30}$$

for some two-form β_2 and ω_1 an harmonic one-form, i.e.

$$e^{-U}(d\ ln\ \psi + A_1 + A_2) = \frac{1}{2}d(e^{-U}) + \delta\beta_2 + \omega_1. \tag{3.31}$$

Therefore,

$$U = \ln\psi, \quad \text{i.e.} \quad \rho = \psi^2 vol_g, \tag{3.32}$$

is the invariant density associated to $H(g,\psi,A_1,A_2)$, with

$$A_1 = \frac{\delta\beta_2}{\psi^2}, \tag{3.33}$$

where β_2 is a smooth two-form on M, and

$$A_2 = \frac{\omega_1}{\psi^2}, \tag{3.34}$$

is an harmonic one-form on M such that ω_1 is as well an harmonic one-form on M; therefore A_2 and ω_1 satisfies (3.21), or what is the same, satisfy (3.22). Conversely, if U satifies (3.30) with some β_2 and an harmonic one-form ω_1, then

$$H(g,\psi,A_1,A_2)^\dagger(e^{-U}) = \delta(e^{-U}(grad\ ln\ \psi + \hat{A}_1 + \hat{A}_2) - \frac{1}{2}d(e^{-U}) = \delta(\delta\beta_2 + \omega_1) = 0,$$

and hence $e^{-U}vol_g$ is an invariant measure; here we have used the facts that $\delta^2 = 0$, and ω_1 is co-closed. Furthermore, the de Rham-Kodaira-Hodge decomposition of the trace torsion one-form Q, is given by

$$Q = d\ ln\ \psi + 1/\psi^2(\delta\beta_2 + \omega_1), \tag{3.35}$$

with β_2 a smooth two-form, and ω_1 an harmonic one-form, both defined on M.

Therefore, we have proved that the equilibrium density ρ is $\psi^2 vol_g$, which we shall further assume always to be normalized, so that

$$\int \psi^2 vol_g = 1,$$

It is important to remark that ρ describes the **final** state of the system. Indeed, one can prove that the transition density $p_\tau^\nabla(x,y)$ tends to $\rho(y)$, for $\tau \to \infty$, and x in a compact set.

It is quite remarkable that the condition of existance of a stationary solution of the Fokker-Planck equation (3.27) leads to a decomposition of the trace-torsion in which there appears two potentials, one of which is harmonic, which are further normalized by $1/\psi^2$. If we allow ψ to have a non-null node set $N(\psi) = \{x \in M : \psi(x) = 0\}$, then these two potentials, $\delta\beta_2/\psi^2$ and ω_1/ψ^2 become singular at $N(\psi)$, giving rise to non-zero monopole charge. The corresponding electromagnetic fields are

$$d(\frac{\delta\beta_2}{\psi^2}) = \frac{1}{\psi^2} d\delta\beta_2 + d(\frac{1}{\psi^2}) \wedge \delta\beta_2$$

$$= 1/\psi^2 (d\delta\beta_2 - 2d\ ln\psi \wedge \delta\beta_2)). \tag{3.36}$$

Taking in account that

$$d\delta\beta_2 = -(\delta d\beta_2 + \triangle\beta_2), \tag{3.37}$$

we obtain by substitution of (3.37) on (3.36) that

$$d(\frac{\delta\beta_2}{\psi^2}) = \frac{-1}{\psi^2}(\triangle\ \beta_2 + \delta\beta_2 + 2d\ ln\ \psi \wedge \delta\beta_2). \tag{3.38}$$

For the harmonic one-form $A_1 = \omega_1/\psi^2$, we have an identical expression as (3.36) substituting ω_1 instead of $\delta\beta_2$; yet, since ω_1/ψ^2 is closed, its electromagnetic two-form field given by $d(\omega_1/\psi^2)$ reduces to

$$dA_2 = d(\frac{\omega_1}{\psi^2}) \equiv 0. \tag{3.39}$$

We remark that the expression (3.39), in account that ω_1 is closed can be written equivalently in the form

$$d\ ln\ \psi \wedge A_2 = 0. \tag{3.39'}$$

We call $A_2 = \omega_1/\psi^2$ the **Aharonov – Bohm potential**; it has a zero field. Finally, the fact that A_2 is co-closed, can be expressed in the form of the conservation equation

$$div_g \hat{A}_2 = -\delta A_2 = 0. \tag{3.40}$$

We note that in (3.40) we see the coupling of the metric to A_2. In fact, the fact that A_2 is harmonic, which is equivalent to equations (3.39, 3.40), implies a coupling of the Aharonov-Bohm potential to the Ricci **metric** curvature. Indeed,

$$\triangle A_2 = trace((\nabla^g)^2)A_2 - R^\beta_\alpha(A_2)_\beta dx^\alpha = 0, \tag{3.41}$$

with

$$R^\beta_\alpha = R_{\mu\alpha}{}^{\mu\beta}$$

Equation (3.41) is the sourceless **de Rham equation** of Misner, Thorne and Wheeler; cf. p.568, [28]. These authors, after de Rham [22], recognize that this equation is indeed more general (and undoubtedly, intrinsic) than the one obtained by the naive prescription of application of the principle of equivalence in the form of the minimal coupling rule of substitution of partial derivatives by Levi-Civita covariant derivatives, without regarding as fundamental the fact that operators (not just derivatives) have to covariantly written. Evidently, if the metric is the Minkowski metric (then $R^\beta_\alpha \equiv 0$), equation (3.40) reduces to the formal implementation of this rule. The same can be said about equation (3.38) whose first term can be explicited to show that β_2 also couples to the metric Ricci curvature. The same coupling of the metric is evident in the expression $\delta\beta_2$, which we shall call the Coriolis potential. We still supplement the equation (3.38) with the Maxwell equation

$$d(*\beta_2/\psi^2) = J_{mag}, \tag{3.42}$$

where J_{mag} denotes the magnetic current three-form.

This has profound implicancies, as to the gauge structure of the present theory. It appears from our formulation, that it is wrong to derive the field theory for g, ψ, A_1, A_2, or still of $g, \psi, \omega_1, \beta_2$ from a Einstein-Cartan-Yang-Mills gauge theory, or more strongly still as just shown, from a Einstein-Yang-Mills gauge theory. (For the non-general applicability of the minimal coupling rule $p \to p - A$ we refer the reader to the authorative treatment of [31]). Clearly, we have given above the field equations for A_1 and A_2: the first is a co-closed one-form, while the second is harmonic; equivalently, we have given the field equations for β_2 and ω_1; or in other words, they β_2 and ω_1 can be discoupled from ψ (this was noticed in [21]), away of the node set of ψ, while they couple to the metric or to the Ricci curvature. Therefore, we are left with the problem of giving the field equations which relate g and ψ. This will be the presented in the second article contributing to these Proceedings. As a final remark, we stress the fact that the Weitzenbock term which expresses the coupling of the electromagnetic potential to the Ricci curvature, plays a fundamental role in the recent theory of monopoles and four-manifolds structures due to Witten and co-workers. In fact, in this theory, the existance of the solutions to Witten's monopole equations, depend essentially on the sign of the metric scalar curvature. [30].

The expressions (3.36) for A_1, A_2 are identical to the modified electromagnetic potential and fields given by the coupling of the usual electromagnetic potentials to the trace-torsion exact component $d\ ln\ \psi$ appearing in the Santilli-Gasperini Lie-isotopic gauge theory [8,21]. Yet, our de Rham-Kodaira-Hodge decomposition of Q has lead us to a deeper characterization of the electromagnetic potential than the usual treatments allow. It is remarkable still, that these modified expressions appear here not as a result of the coupling of the electromagnetic field to torsion, which demands the well known modifications of the minimal coupling rules, but as a consequence of setting the drift torsion vector field to yield an equilibrium density of the Markovian semigroup.

Finally, we shall introduce the probability vector associated to the RCW diffusion. Consider the vector field

$$J_\tau := p_\tau^\nabla b - \frac{1}{2} grad\ p_\tau^\nabla \tag{3.43}.$$

Then, the Fokker-Planck equation can be written as

$$\frac{\partial p_\tau^\nabla}{\partial \tau} + div_g J = 0, \tag{3.44}$$

which in the stationary case, yields the probability vector field

$$J = \rho b - \frac{1}{2} grad\ \rho, \tag{3.45}$$

where

$$b = 1/2 grad\ ln\ \rho + \hat{A}/\rho, \tag{3.46}$$

i.e.

$$b^\alpha = 1/2 g^{\alpha\beta}(\partial_\alpha ln\ \psi + 1/\psi^2 A_\beta) \tag{3.46'}$$

with A given by

$$A = \delta\beta_2 + \omega_1.$$

Therefore, J reduces to

$$J = \hat{A}, \tag{3.48}$$

Then, J is a conserved probability vector field, since

$$div_g J = -\delta\tilde{J} = -\delta\hat{A} = 0. \tag{3.49}$$

Note that only in the case in which we set $A \equiv 0$, we have a null probability vector. This will be of importance when we discuss the notion of detailed balance, which introduces the figure of microscopic reversibility, which is usually introduced in statistical mechanics through the assumption of a hamiltonean function. We should remark here that we have

not even mentioned in our derivation of the equilibrium measure the usual assumption of a hamiltonean function. We see no point in assuming the existance of a hamiltonean and further assume reversibility for later describing non-equilibrium statistical mechanics.

Finally, we are in condition of observing that in the covariant setting described in this section, we have the following identities between the coefficient functions in the Kramers-Moyal expansion, and the geometrical structures introduced in this section. They are,

$$\Lambda^{\alpha} = b^{\alpha}, \tag{3.50}$$

or in other words, the first moments are given by the drift trace-torsion vector- field described in $(3, 46, 46')$, and

$$\Lambda^{\alpha\beta} = g^{\alpha\beta}. \tag{3.51}$$

Therefore, the metric describes the second order fluctuational terms, yet the drift vector field incorporates these fluctuations by definition due to the elementary fact that it is the conjugate vector field to a one-form, Q. This has appeared from the intrinsic formalization of the theory. If this invariance by diffeomorphisms of the manifold M is not respected,it seems that the fluctuations are independent of the first order contributions given by the drift.

Therefore, the drift is not to be considered a purely classical term. We shall return to this when presenting the non-linear H-theorem of this theory.

II.b Riemann-Cartan-Weyl Diffusions

By embedding M on R^d, with $d \leq 2n+1$, we can obtain a section Y (at least locally Lipschitz, or still, satisfying Sobolev regularity conditions) of $L(R^d, TM)$, so that if Y^* denotes the dual section of $L(TM, R^d)$, then for all $x \in M$,

$$g(x) = Y(x)Y^*(x). \tag{3.52}$$

Given an orthonormal basis $\{e_i, i = 1, \ldots, n\}$ of R^d, we may define vector fields

$$Y_i(x) = Y(x)(e_i). \tag{3.53}$$

Taking Y to be smooth, we can define the second-order differential operator

$$L_Y^2 = (Y_i)^2. \tag{3.54}$$

For L as in (1) and Y locally of the form

$$Y_i(x) = Y_\beta^\alpha(x)\partial_\alpha, \tag{3.55}$$

then, locally

$$L_Y^2(x) = g^{\alpha\beta}\partial_\alpha\partial_\beta + Y_\alpha^\beta(x)\partial_\beta Y_\alpha^\gamma(x)\partial_\gamma. \tag{3.56}$$

If we take the vector field on M given by

$$X_0^Y(x) = b(x) - \frac{1}{2}Y_\alpha^\beta\partial_\beta Y_\alpha^\gamma(x)\partial_\gamma \tag{3.57}$$

with $b = \hat{Q}$, then

$$L = \frac{1}{2}L_Y^2 + X_0^Y. \tag{3.58}$$

This decomposition, while still invariant by diffeomorphisms of M, it depends essentially on the choice of the "square root" Y of σ.

For an arbitrary Riemann-Cartan connection ∇ as in (3.15), we consider its associated Levi-Civita connection ∇^g; with the choice of Y sufficiently regular, we have the following decomposition of L:

$$X_0^Y = X_0(\nabla) - S(\nabla^g, Y), \tag{3.59}$$

and

$$\frac{1}{2}L_Y^2 = \sigma(\nabla) + S(\nabla^g, Y) \tag{3.60}$$

with

$$S(\nabla^g, Y) = \frac{1}{2}tr(\nabla^g Y(Y(.)(.)), \tag{3.61}$$

where $\nabla^g Y$ is the covariant derivative of Y viewed as a section of $L(TM, L(R^d, TM))$. The vector field $S(\nabla^g, Y)$ is the Stratonovich correction term. It is essential for the transformation of the representation of the process according to the Stratonovich midpoint rule which gives the Stratonovich stochastic differential calculus with transformation rules similar to the ordinary differential calculus, to the Ito representation (prepoint rule).

If we take for ∇ the RCW connection (3.15), since $\sigma(\nabla) = \frac{1}{2}tr(\nabla^2) = H(g, \psi, A_1, A_2)$, then,

$$X_0^Y = -S(\nabla^g, Y), \tag{3.62}$$

and

$$\frac{1}{2}L_Y^2 = \frac{1}{2}\triangle_g + b + S(\nabla^g, Y). \tag{3.63}$$

Given a Fokker-Planck operator, one can characterize the associated Markovian semigroup through the heat kernel, or still, through a stochastic differential equation; for the non-covariant formulation of this, see [2]. In particular, for the Markovian semigroups

determined by $H(g, \psi, A_1, A_2)$, the associated stochastic differential equation for the random continuous curves $\{B_\tau : \tau \geq 0\}$ are

$$dB_\tau = Y(B_\tau) \circ dW_\tau + b(B_\tau) + S(\nabla^g, Y))(x_\tau)d\tau, \tag{3.64}$$

in the Stratonovich representation, or equivalently, in the form of the Ito representation as

$$dB_\tau = Y(B_\tau)dW_\tau + b(B_\tau)d\tau, \tag{3.65}$$

with b as in (3.46) and $\{W_\tau, \tau \geq 0\}$ is a mean 0 Brownian process on R^d: $< W_\tau >= 0$ and $< W_\tau^r W_\tau^s >= \delta_{rs}\tau$. This is the covariant formulation of the treatment given in [2].

It is remarkable that the solution flow of (3.65) can be formulated as a stochastic process on the space of tempered distributions on M provided with the Gaussian measure. This means that the quantum diffusions are quantum fields [29]. The description of this aspect would carry us away of the main objectives of this article, and thus will be presented elsewhere.

IV. DETAILED BALANCE AND MICROSCOPICAL REVERSIBILITY

The case of a vanishing probability current, i.e. $b = \textit{grad ln } \psi$, is a particular case of the physical phenomenae of detailed balance. Roughly speaking, a Markov process satisfies detailed balance if in the stationary situation each possible transition balances with the reversal transition.

The next example illustrates in this point, yet on thinking on a physical system, more specifically, a gas, whose evolution we study in phase space. Consider thus a gas of particles with positions r and velocities v. A transition corresponds to the case of a particular time τ with position and velocity r and v, respectively, having acquired at a later time $\tau + s$ a position and velocity r' and v'. The probability density of this transition is the joint probability density $p_{\tau,\tau+s}(r, v; r', v')$. We may represent this transition in the form

$$(r, v, \tau) \rightarrow (r', v', \tau + s).$$

The reversed transition is

$$(r', -v', \tau) \rightarrow (r, -v, \tau + s);$$

it corresponds to the time reversed transition and requires that the velocities to be reversed because the motion from r' to r, is in the opposite direction from that from r to r'. The probability density for the reversed transition is the joint probability density $p_{\tau,\tau+s}(r', -v'; r, -v)$.

The principle of detailed balance is that at the stationary state, these two previous joint probabilities densities coincide

$$p_{eq}(r', v', \tau; r, v, 0) = p_{eq}(r, -v, \tau; r', -v', 0).$$

Thus, for a Markov process we may write

$$p(r', v', \tau | r, v, 0) p_{eq}(r, v) = p(r, -v', \tau | r', -v', 0) p_{eq}(r', -v').$$

In its general form, detailed balance is formulated in terms of the symmetry behaviour of arbitrary variables x_i, which under time reversal transform to the reversed variables, according to the rule

$$x_i \rightarrow \epsilon_i x_i, \quad \text{(no summation)} \tag{4.1}$$

and

$$\epsilon_i = +1, \epsilon = -1, \tag{4.2}$$

respectively, depending on whether the variable is even or odd respectively, under time reversal. In the situation of the previous example, r is even, v is odd.

Then, in the general formulation, detailed balance is written as

$$p_{eq}(x, s + \tau; x', s) = p_{eq}(\epsilon x', s + \tau; \epsilon x, s). \tag{4.3}$$

Here we have denoted ϵx for $(\epsilon_1 x_1, \epsilon_2 x_2, \ldots)$. Notice that setting $\tau = 0$ in the definition, we get

$$\delta(x - x') p_{eq}(x') = \delta(\epsilon x - \epsilon x') p_{eq}(\epsilon x). \tag{4.4}$$

Yet, the two delta functions are equal since only sign changes are involved. Hence,

$$p_{eq}(x) = p_{eq}(\epsilon x). \tag{4.5}$$

Rewriting now the definition in terms of conditional probabilities we have

$$p(x, \tau | x', 0) p_{eq}(x') = \delta(\epsilon x - \epsilon x') p_{eq}(\epsilon x). \tag{4.6}$$

To further elaborate on the physical implications of detailed balance, one can show that for detailed balance we have the following necessary and sufficient conditions on the coefficients Λ^{α} and $\Lambda^{\alpha\beta}$, for any α, β; these are (see [2]):

$$\epsilon_{\alpha} \Lambda^{\alpha}(\epsilon x) p_{eq}(x) = -\Lambda^{\alpha}(x) p_{eq}(x) + \partial_{\beta} \Lambda^{\alpha\beta} \; p_{eq}(x), \tag{4.7}$$

and

$$\epsilon_{\alpha} \epsilon_{\beta} \Lambda^{\alpha\beta}(\epsilon x) = \Lambda^{\alpha\beta}(x). \tag{4.8}$$

We wish to depart form this phase-space description and concentrate on even space-time variables x. Note that from (4.5) it follows that

$$< x >_{\text{stat}} = \epsilon < x >_{\text{stat}},$$

and hence all odd variables, in particular velocities, have zero stationary means.

Thus, it becomes clear that a detailed balance approach associated to phase-space diffusions, is pointless. This is quite remarkable, as the hamiltonean approach to statistical mechanics, is the one applied to the formulations of linear non-equilibrium statistical mechanics (Prigogine's work [4,19]), as well as the contemporary non-linear extension by Stratonovich's [2]. In Stratonovich's approach, it is considered to be an axiom of the theory.

The detailed balance hamiltonean approach in statistical mechanics is the remora of Botzmann's assumption of symmetry of the collision term in his kinetic equation. The history of physics records the criticisms by Zermelo, Loschmidt and Poincaré [25,26,27] to Boltzmann's approach to irreversibility, in which by assuming the reversibility on the velocities, intended to derive that the entropy of the system monotonically increases to its maximum attained by p_{eq}, Maxwell's distribution. We see no point in persuing such an approach which till today has not found an alternative other than the Onsager-Machlup lagrangian formulation of the Feynman path-integral representation of the transition density [16,18]. We shall summarily introduce this alternative formulation, from the point of view of the geometrical approach developed in this article, in the complementary article to the present one, in these Proceedings.

Our theory, stems from space-time (space) infinitesimal group symmetries. For the case of space-time provided with a Riemannian metric, the time coordinate can be assumed to be even, if and only if the Coriolis potential $\delta\beta_2$ and the Aharonov-Bohm potential vanishes completely, so that it generates a conserved flow, as we shall see next.

So, we are interested in even variables. For even variables, detailed balance conditions yield the trivial equation

$$\Lambda^{\alpha\beta}(x) = \Lambda^{\alpha\beta}(x), \tag{4.9}$$

and

$$\Lambda^{\alpha}(x)p_{eq}(x) = -\Lambda^{\alpha}(x)p_{eq}(x) + \partial_\beta \Lambda^{\alpha\beta}p_{eq}(x). \tag{4.10}$$

We know already from (3.48) that $\Lambda^{\alpha\beta} = g^{\alpha\beta}$, while Λ^{α} is determined by (3.43, 47). It is immediate that the covariant expression of (4.10) is

$$Q(x)p_{eq}(x) = 1/2d\ p_{eq}(x), \tag{4.11}$$

or what is the same (if $g = (g^{\alpha\beta})$ is non-singular) that

$$b(x)p_{eq}(x) = 1/2grad\ p_{eq}(x) \tag{4.12}$$

or still, equivalently

$$J = \hat{A} = 0, \tag{4.13}$$

with $A = \delta\beta_2 + \omega_1$. Yet, from the smoothness of β_2 and ω_1 and their orthogonality, we conclude that the detailed balance condition in covariant form, is equivalent to

$$J = 0, \tag{4.14}$$

with

$$\delta\beta_2 \equiv \omega_1 \equiv 0. \tag{4.15}$$

Therefore, the vanishing of **both** the Aharonov-Bohm component of the trace-torsion and the Coriolis potential $\delta\beta_2$, is equivalent to detailed balance.

We want to analyze the relation between detailed balance and microscopic reversibility. Suppose we set the fluctuation components of the process to be identically zero, so that detailed balance is written according to equation (4.10) as

$$\epsilon_\alpha \Lambda^\alpha(ex) = -\Lambda^\alpha(x), \tag{4.16}$$

and the Fokker-Planck equation reduces to the Liouville equation

$$\frac{\partial p(x,\tau;x',\tau')}{\partial \tau} = -\frac{\partial}{\partial x^\alpha}(\Lambda^\alpha(x,\tau)p(x,\tau|x',\tau')). \tag{4.17}$$

The motion of a point $x(\tau)$ is described by the ordinary differential equation determined by the non-covariant drift

$$dx^\alpha(\tau)/d\tau = \Lambda^\alpha(x(\tau)). \tag{4.18}$$

Suppose a solution of this equation passes at y when $\tau = 0$; we write this solution as $q(\tau, y)$; then $q(0, y) = y$. The relation of detailed balance implies that

$$\epsilon q(-\tau, \epsilon y) \tag{4.19}$$

is also a solution of the classical flow for b, and since

$$\epsilon q(-\tau, \epsilon y) = \epsilon^2 y = y,$$

i.e., the initials conditions of both solutions coincide, therefore by uniqueness we have

$$\epsilon q(-\tau, \epsilon y) = q(\tau, y). \tag{4.20}$$

Thefore, for a purely deterministic flow determined by the solution curves of the drift, detailed balance is nothing else than an example of the usual reversibility of classical dynamical systems. Here, in spite that we have not assumed -as customary- the existance of a hamiltonean function, the classical Liouville equation is subsummed in the Fokker-Planck equation. Yet, the problem with equation (4.18) is that has **not** been obtained

in a covariant way by assuming the vanishing of the fluctuational terms , so that the separation of the fluctuation terms described by the metric,which we have set in (4.10) to be identically null, with the trace-torsion one-form, is meaningless. The classical reversibility linked to the flow of $\Lambda = (\Lambda^{\alpha})$ has no intrinsic meaning. The correct covariant form of the detailed balance condition on $\Lambda^{\alpha} = b^{\alpha}$ is:

$$b(x) = -b(x) + grad\ ln\ p_{eq}(x), \tag{4.21}$$

or equivalently

$$b(x) = 1/2d\ ln\ p_{eq}(x) = 1/2grad\ ln\ \rho, \tag{4.22}$$

which as we saw already is equivalent to $J \equiv 0$.

Yet, if in the invariant form of the detailed balance condition (4.15), i.e. an equation between (gradient) vector fields we set the fluctuation term $g_{\alpha\beta} \equiv 0$, we are left with a trivial identity.

There is an intrinsic first-order equation, given by

$$dx(\tau)/d\tau = b(x), \tag{4.23}$$

where $b(x)$ is the trace-torsion vector field (3.43), which not necessarily reduces to its component $1/2.grad\ ln\rho$, yet this equation is deduced from detailed balance in this case in which the trace-torsion reduces to the component derived from the equilibrium density.

Therefore, in our derivation of the detailed balance condition $J \equiv 0$, one can assign an intrinsic meaning to microscopical classical reversibility as associated to the flow of the vector field $grad\ ln\ \psi$ which produces a trivial probability current.

One last comment on this section. Our discussion above in the impossibility of obtaining covariantly the Liouville operator, sets us the further problem of the derivation of the classical perfile of the theory. Our above discussion of the problem of irreversibility linked to the flow of b the non-gradient drift vector field obtained from detailed balance has been useful to identify the origin of irreversibility in the stochastic flow defined by integrating (3.65), in the electromagnetic potentials appearing in the trace-torsion. The setting of the classical theory as derived from the geometry of the fluctuations requires a different analysis which will be presented in the second article contributing to these Proceedings. It is framed in terms of the above mentioned Onsager-Machlup lagrangian for the most probable smooth approximations to the fluctuations.

V. THE CANONICAL REPRESENTATION OF THE MARKOV FUNCTION

We have given in the previous section, an invariant decomposition of the Markov generating function in terms of a Lie-isotopic geometry, and thus linked the first moment of the Markov process with the trace-torsion tensor.

Thus, the role of torsion is essential to the description of fluctuations. In a Markov process the first two moments determine all higher moments, so as observed already the Markov generating function describes all the statistics of the process.

We would like to further elaborate on this, firstly by giving a canonical representation of the Markov generating function of a stationary Markov process, in which the role of torsion can be isolated. This canonical decomposition is essential to formulate a non-linear H-theorem and to see that the trace-torsion plays the central role in the system's approch to macroscopic equilibrium.

To introduce this canonical decomposition of the Markov generating function [1,3], we firstly introduce the notion of infinitely divisible probability densities

V.A. Infinite Divisible Probability Densities

We start with the case of random variables, i.e. for uni-dimensional processes to finally give the decomposition for vector-valued processes.

Let ξ be a random variable, and $p(\xi)$ is its probability density. Then, ξ is called infinitely divisible, if there exists sequences of random $\eta_j, j \in \{1,\ldots,n\}$, n arbitrary large, all having the same probability density $p_n(\eta_j)$,such that

$$\xi = \eta_1 + \ldots + \eta_n. \tag{5.1}$$

Let

$$\Xi(iu) = \int exp(iu\xi)p(\xi)d\xi, \quad \Xi_n(iu) = \int exp(iu\eta)p_n(\eta)d\eta \tag{5.2}$$

Recalling the basic property that the characteristic function of statistically independent random variables equals the product of the characteristic functions of each variable, and further, since the variables have the same probability density and so the same characteristic function, we conclude that the characteristic function of the sum ξ is equal to the power n of the characteristic function $\Xi_n(iu)$, i.e.

$$\Xi(iu) = [\Xi_n(iu)]^n. \tag{5.3}$$

This means that the condition of infinite divisibility of p_ξ reeduces to the fact that $[\Xi(iu)]^{\frac{1}{n}}$ is the characteristic function of a random variable, or equivalently, the inverse

Fourier transform of $[\Xi_n(iu)]^{\frac{1}{n}}$ is a probability density

$$p_n(\eta) = (2\pi)^{-1} \int exp(-iu\eta)[\Xi(iu)]^{\frac{1}{n}}; \tag{5.4}$$

thus, p_η is non-negative and normalized. Indeed, we have

$$\int p_n(\eta)d\eta = \int du(2\pi)^{-1} \int d\eta exp(-iu\eta)[\Xi(iu)]^{\frac{1}{n}}$$

$$= \int \delta(u)[\Xi(iu)]^{\frac{1}{n}} du = [\Xi(0)]^{\frac{1}{n}} = 1^{\frac{1}{n}} = 1.$$

Therefore,we have proved that p_η is normalized. We have to prove that

$$\int exp(-iu\eta)[\Xi(iu)]^{\frac{1}{n}} du \geq 0,$$

for any n and η, so that $p_n(\eta)$ is infinitely divisible. Yet, it is known (cf. [1,3]) that a probabilitity density p_ξ of a random function ξ, satisfying the condition

$$\int \xi^2 p(\xi)d\xi < \infty, \tag{5.5}$$

is infinitely divisible if and only the integral representation of its characteristic function

$$ln\ \Xi(iu) = -i\gamma u + \int_\infty^\infty [exp(iuz) - 1 - iuz]z^{-2}H(z)dz, \tag{5.6}$$

is valid. Here $H(z)$ is a non-negative integrable function (which can have singularities of the delta type), and γ is a real constant.

The previous results for a random variable can be generalized to a vector-valued random variable, $\xi = (\xi_1, \ldots, \xi_n)$. In this case η, u and $z = (z_1, \ldots, z_n)$ are vectors. So, $uz = u_\alpha z_\alpha$, $z^2 = (z_\alpha)^2$.

Let us consider now a stochastic process $y(\tau) = (y_1(\tau), \ldots, y_n(\tau))$ which we assume to be a stationary process. We shall say that this process has independent increments if for any $\tau_1 < \ldots < \tau_n$, the increments $y(\tau_2) - y(\tau_1), y(\tau_3) - y(\tau_2), \ldots, y(\tau_n) - y(\tau_{n-1})$, are statistically independent.

It is immediate that for a stationary process with independent increments, the probability density $p(y(\tau') - y(\tau))$ is infinitely divisible, for $\tau' > \tau$. Indeed, if we set $s = (\tau' - \tau)/n$, the increment can be represented as the sum of independent increments

$$y(\tau') - y(\tau) = [y(\tau + s) - y(\tau)] + [y(\tau + 2s) - y(\tau + s)] + \ldots + [y(\tau') - y(\tau' - s)], \tag{5.7}$$

for arbitrary n. Note that these increments have the same probability density due to the stationarity of the process. Therefore, if each summand has a finite second moment, the condition for the representation (5.6) is valid, ie., the characteristic function given by

$$\Xi_\tau(iu) =< exp(iu[y(\tau+t) - y(\tau)]) >, \tag{5.8}$$

can be represented as

$$ln\ \Xi_\tau(iu) = i\gamma_\tau u + \int (exp(iuz) - 1 - iuz)z^{-2}H_\tau(z)dz \tag{5.9}$$

where $H_\tau(z)$ is a non-negative integrable function. This representation for the characteristic function, we shall call the **canonical representation**.

Now, the condition of independence of increments implies that the characteristic function satisfies

$$\Xi_{\tau_1+\tau_2}(iu) = \Xi_{\tau_1}(iu) + \Xi_{\tau_2}(iu) \tag{5.10}$$

which implies that $ln\ \Xi_\tau(iu)$ is proportional to τ

$$ln\ \Xi_\tau(iu) = \tau\phi(iu) \tag{5.11}$$

Comparison of (5.11) with the canonical representation (5.9) yields

$$\gamma_\tau = \gamma_1\tau, H_\tau(z) = H_1(z)\tau. \tag{5.12}$$

whre $H_1(z)$ is necessarily a non-negative integrable function.

The crucial fact is that a process with independent increments is a Markov process, yet its characteristic function looses its dependence in y'. Then, we proceed to the master equation. From (2.20), after passing to the limit we get

$$\Phi(v) = \phi(v), \tag{5.13}$$

so that the master equation is

$$\frac{\partial p_\tau(y|y")}{\partial\tau} = \phi(-\frac{\partial y}{\partial y})p_\tau(y|y"). \tag{5.14}$$

Then, from the relations (5.11), we can write

$$\Phi(v) = \phi(v) = \gamma^1\ ^\alpha v_\alpha + \int_{-\infty}^{\infty}\{exp(vz) - 1 - vz\}z^{-2}H_1(z)dz,\ Re(v) = 0, \tag{5.15}$$

yet, in comparing with ther Kramers-Moyal expansion, $\gamma^{1\ \alpha} = \Lambda^{\alpha}$,for every α, and therefore,

$$\Phi(v) = \phi(v) = \Lambda^{\alpha} v_{\alpha} + \int_{-\infty}^{\infty} \{exp(vz) - 1 - vz\} z^{-2} H_1(z) dz. \tag{5.16}$$

One can see clearly the drift trace-torsion vector field (cf. (3.50)) in the first term of this representation. This canonical representation will appear to be fundamental in the following.

Of course, what matters here is the applicability of this representation to a general stationary Markov process. In this case, the probability densities of the increments $y(\tau') - y(\tau)$, $\tau' > \tau$, are not strictly speaking infinitely divisible. Yet, for small increments $s = \tau' - \tau$, the infinitely divisibility of the increments is valid ; cf. p.56 [1]. This is a crucial fact.

VI. THE KINETIC POTENTIAL AND ITS IMAGE

In this section, we shall consider a stationary Markov process $B(\tau) = (B^1(\tau), \ldots, B^n(\tau))$ which we may think as the thermodynamic equilibrium fluctuations. This processes is characterized by the coefficient functions

$$\Lambda^{\alpha_1 \ldots \alpha_n} = lim_{\tau \to 0} \frac{1}{\tau} < \triangle B^{\alpha_1} \ldots \triangle B^{\alpha_n} >_B \tag{6.1}$$

where,

$$\triangle B = B(\tau_1 + \tau) - B(\tau_1), \quad B = B(\tau_1). \tag{6.2}$$

If this limit exists, we can estimate the coefficients for $\tau > 0$,

$$< \triangle B^{\alpha_1} \ldots \triangle B^{\alpha_n} >= \Lambda^{\alpha_1 \ldots \alpha_n}(B)\tau + o(\tau). \tag{6.3}$$

Inverting the formula that links correlators with moments, we conclude that the correlators have similar estimates:

$$< \triangle B^{\alpha_1}, \ldots \triangle B^{\alpha_n} >_B = \Lambda^{\alpha_1 \ldots \alpha_n}\tau + o(\tau) \tag{6.4}$$

If we assume that the correlators are polynomials of the Boltmann constant k

$$< \triangle B^{\alpha_1}, \ldots, \triangle B^{\alpha_n} > k^{n-1}, \tag{6.5}$$

we then introduce the nonequilibrium kinetic potential $V(y, B)$:

$$\Lambda^{\alpha_1 \ldots \alpha_n}(B) = (kT)^{n-1} [\frac{\partial^n V(y, B)}{\partial y_{\alpha_1} \ldots \partial y_{\alpha_n}}] | y = 0, 1 \leq n \tag{6.6}$$

Then, we can write the kinetic potential as the Taylor series expansion

$$V(y,B) = \sum_{n=1}^{\infty} \frac{1}{m!} \beta^{n-1} \sum_{\alpha_1 \ldots \alpha_n} \Lambda^{\alpha_1 \ldots \alpha_n}(B) y_{\alpha_1} \ldots y_{\alpha_n}, \quad \beta = \frac{1}{kT} \tag{6.7}$$

The Markov generating function and the kinetic potential are related by

$$\Phi(v,B) = \beta . V(kTv, B), \tag{6.8}$$

so that the following master equation

$$\frac{\partial p(B)}{\partial \tau} = N_{\partial,B} \beta . V(-kT . \frac{\partial}{\partial B}, B) p(B). \tag{6.9}$$

is valid.

Let us introduce the free energy function $F(B)$ as

$$p_{eq}(B) = exp(-\beta . F(B)). \tag{6.10}$$

i.e.,

$$\psi^2(B) = exp(-\beta . F(B)). \tag{6.11}$$

We further introduce the probability density

$$p_z(B) = C . exp(-\beta(F(B) - z.B)). \tag{6.12}$$

Here we have introduced hypothetical thermodynamic forces $z = (z_\alpha)$, which couple couple to the fluctuational process B through the internal product $z.B = z_\alpha B^\alpha$ which we shall describe further below. Substituting (6.1) into (6.7) we obtain

$$V(y,B) = lim_{\tau \to 0} \frac{1}{\tau} \sum_{m=1}^{\infty} \frac{1}{m!} \beta^{m-1} < (y_\alpha \triangle B^\alpha)^m >_B = 1/\beta lim_{\tau \to 0} < exp(\beta y_\alpha \triangle B^\alpha) - 1 >_B \tag{6.13}$$

Let us compute the matrix of second derivatives of V

$$V_{\alpha\beta} := \frac{\partial^2 V(y,B)}{\partial y^\alpha \partial y^\beta} = \beta lim_{\tau \to 0} \frac{1}{\tau} < \triangle B^\alpha \triangle B^\beta exp(\beta y \triangle B) >_B . \tag{6.14}$$

Let us prove that this matrix is positive definite: $V_{\alpha\beta} a^\alpha a^\beta \geq 0$, for any vector $a = (a^\alpha)$. Indeed,

$$V_{\alpha\beta} a_\alpha a_\beta = \beta lim_{\tau \to 0} \tau^{-1} < (a_\alpha \triangle B^\alpha)^2 exp(\beta y_\alpha \triangle B^\alpha) >_B \ \geq 0. \tag{6.15}$$

Consequently, $V(y,B)$ is a convex function of y. From the convexity of V with respect to y, we conclude that the kinetic potential of the fluctuations is non-negative:

$$V_{fluc}(y,B) = V(y,B) - \Lambda^{\alpha}(B)y_{\alpha} = \sum_{m=2}^{\infty} \frac{1}{m!}\beta^{m-1}\Lambda^{\alpha_1}\ldots\alpha_n y_{\alpha_1}\ldots y_{\alpha_n} \geq 0. \qquad (6.16)$$

We shall now proceed to introduce the **image** of the kinetic potential $R(y,z)$ which is defined by

$$R(y,z) := \int V(y,B)p_z(B)dB. \qquad (6.17)$$

Consider now the equilibrium density $\rho_{eq} = \psi^2$, i.e. the time independant solution of the Fokker-Planck equation; we naturally ask ourselves what happens when we rescale p_{eq} to p_z ? The obvious fact is that the operator of multiplication by $exp(\beta xB)$ does not commute with $\frac{\partial}{\partial B}$; instead, we have

$$exp(\beta zB)f(\frac{\partial}{\partial B} + \beta z) = f(\frac{\partial}{\partial B})exp(\beta zB). \qquad (6.18)$$

Then, we have

$$exp(\beta zB)N_{\partial B}V(-kT\frac{\partial}{\partial B}) = N_{\partial,B}V(-kT\frac{\partial}{\partial B} + z, B)exp(\beta zB). \qquad (6.19)$$

Inserting this in the Fokker-Planck equation multiplied by $exp(\beta zB)$ and further integrating with respect to B from $-\infty$ to $+\infty$, we find that all terms that have $\frac{\partial}{\partial B}$ in the argument of V vanish. Thus, the integral that remains is

$$\int V(z,B)exp(\beta zB)p_{eq}(B)dB = 0, \qquad (6.20)$$

i.e.

$$< V(x,B) >_z = 0, \quad \text{with} \quad < \; >_z = \int (\;)p_z(B)dB, \qquad (6.21)$$

or still,

$$R(z,z) = 0, \quad \text{for all } z. \qquad (6.22)$$

Due to the smallness of the fluctuations of B, the probability density $p_z(B)$ is concentrated near it maximum point $A(z)$. Therefore, from (6.17) we obtain the approximate expression

$$R(y,z) = V(y,A(z)) + O(k). \qquad (6.23)$$

This will have profound implications in the formulation of the non-linear H-theorem.

Substituting the expansion (6.7) in the definition (6.17) of the image of the kinetic potential we get the expansion

$$R(y,z) = \sum_{m=1}^{\infty} 1/m!\beta^{m-1}\kappa^{\alpha_1\ldots\alpha_m}(z)y_{\alpha_1}\ldots y_{\alpha_m}, \tag{6.24}$$

where

$$\kappa^{\alpha_1\ldots\alpha_m}(z) = \int \Lambda^{\alpha_1\ldots\alpha_m}(B)p_z(B)dB. \tag{6.25}$$

Note that from (6.4,6.25) we conclude that we can also write

$$\kappa^{\alpha_1\ldots\alpha_m}(z) = lim_{\tau\to 0}1/\tau < \triangle B^{\alpha_1}\ldots\triangle B^{\alpha_m} >_z . \tag{6.26}$$

Substituting (6.26) in (6.24) we obtain the formula

$$R(y,z) = kTlim_{\tau\to 0}1/\tau < exp(\beta yB) - 1 >_z, \tag{6.27}$$

Analogously to V, one can prove that $R(y,z)$ is convex as a function of the variable y, and the fluctuational part of $R(y,z)$ defined by

$$R_{fluc}(y,z) = R(y,z) - \kappa^{\alpha}(z)y_{\alpha}, \tag{6.28}$$

verifies

$$R_{fluc}(y,z) \geq 0. \tag{6.29}$$

It is immediate that the image of the kinetic potential admits the integral representation derived from the canonical representation

$$R(y,z) = \kappa^{\alpha}(z)y_{\alpha} + \int_{-\infty}^{+\infty} f(s)s^{-2}G(s,z)ds,$$

where

$$G(s,z) = \int H(kTs,B)p_z(B)dB.$$

Until now, we have given the formulation in an Euclidean setting. In the covariant formulation of the theory, we have still the problem of giving sense to the expressions of the form $exp(\beta zB)$, or still, $\kappa^{\alpha}y_{\alpha}$, i.e. to physically express the coupling of thermodynamic forces to the fluctuational Brownian processes. We have to describe the metric through which these couplings are produced. In other words, to describe which is the small-scale representation of the large-scale cosmological fluctuations (described by the

metric g), when one changes from the cosmological scale to the local scale by rescaling the equilibrium density $C.\psi^2$ by multiplying it by

$$exp(\beta z_\alpha B^\alpha);$$

cf. (6.18).

We shall assume that the internal product meant in this local rescaling is given by the conformal metric deduced from the cosmological scale, given by

$$\psi^2 g_{\alpha\beta},$$

so that

$$z_\alpha B^\alpha = \psi^2 g(z, B). \tag{6.30}$$

Note that the coupling of (6.30) is non-negative. This assumption, which we shall not discuss further in this work, has profound implications on the balance of the local and global tendency to equilibrium, which is regulated by the potential acting of the fluctuation processes; this is the Bohmian quantum potential which is the potential which depends non-linearly on the invariant distribution of the system [17, 18]. We shall only discuss in the second article to these Proceedings, the relation between the Bohm potential produced by the equilibrium density and the global fluctuation metric.

VII.THE GEOMETRY OF THE FLUCTUATION-DISSIPATION RELATIONS AND THE NON-LINEAR H-THEOREM

Assume a fluctuational homogeneous Markov process described by the stochastic differential equation (3.65)

$$dB_\tau = b(B(_\tau))d\tau + Y(B_\tau)dW_\tau,$$

where

$$b = grad\ ln\ \psi + 1/\psi^2(\delta\beta_2 + \omega_1).$$

is as in (3.50). Y is a square root of g (cf. (3.52)). We recall that in this covariant description

$$\Lambda^\alpha = b^\alpha$$

and

$$\Lambda^{\alpha\beta} = g^{\alpha\beta};$$

also, $p_{eq} = \psi^2 = exp(-kF)$, where F is the free-energy function.

We have

$$\kappa^\alpha(z) =< b^\alpha >_z = \int b^\alpha(B)\psi^2(B)exp(\beta < z, B >)dB. \tag{7.1}$$

Assume that the means

$$A^\alpha(\tau) =< B^\alpha_\tau > \tag{7.2}$$

satisfy the non-linear fluctuation-dissipation equations

$$dA^\alpha/d\tau \equiv d < B^\alpha_\tau > /d\tau = \phi^\alpha(A(\tau)) \tag{7.3}$$

Yet, since

$$d < B_\tau > /d\tau =< dB_\tau >= b(B_\tau) \tag{7.4}$$

where the first identity follows from the Ito formula and the second identity follows from (3.65), the fluctuation-dissipation equations are

$$b(B_\tau) = \phi(A(\tau)), \quad \text{where} \quad A(\tau) =< B_\tau > . \tag{7.5}$$

It is remarkable that in this last form, the fluctuation-dissipation equations have no relation with proper-time evolution of the fluctuation means.

Now, from (6.26) for $m \geq 1$ we get

$$< dB^\alpha_\tau >_z = \kappa^\alpha(A). \tag{7.6}$$

Yet, since $p_z(B)$ has a maximum at $A(z)$, we have that

$$< dB^\alpha_\tau >_z =< \phi^\alpha(A) >_z, \tag{7.7}$$

equals to

$$\phi^\alpha(A(z)). \tag{7.8}$$

Therefore, the fluctuation-dissipation equations assume the form

$$dA^\alpha/d\tau = \phi^\alpha(A(z)) = \kappa^\alpha(z(A)), \tag{7.9}$$

or still, in rewriting the non-evolutionary expression (7.5), we have

$$b^\alpha(B_\tau) = \kappa^\alpha(z). \tag{7.9'}$$

The relation between z and A is that the z are assumed to be thermodynamic forces conjugate to A, i.e.

$$z = \nabla_A F = \frac{\partial F(A)}{\partial A} \tag{7.10}$$

Note that F being a scalar, in the definition of z, the covariant derivative can be either the Levi-Civita connection or the RCW connection. From (6.22) we have

$$R(z,z) = z_\alpha \kappa^\alpha(z) + R_{fluc}(z,z) = 0,$$

then, from (6.30) we conclude that

$$z_\alpha \kappa^\alpha(z) = -R_{fluc}(z,z) \leq 0, \tag{7.11}$$

and therefore

$$z_\alpha \kappa^\alpha(z) \leq 0. \tag{7.12}$$

Finally, taking in account (7.9, 7.11), we have

$$dF(A)/d\tau = \frac{\partial F(A)}{\partial A}\frac{dA}{d\tau} = z_\alpha dA^\alpha/d\tau = z_\alpha \kappa^\alpha(z) \leq 0. \tag{7.13}$$

Therefore

$$dF(A)/d\tau \leq 0, \tag{7.14}$$

which can be written as

$$-\beta \frac{d\ ln\ \psi^2}{d\tau}(< B_\tau >) \leq 0. \tag{7.15}$$

From (7.13, 6.26, 6.28) it follows that the derivative of the free energy

$$dF/d\tau = -kT lim_{\tau \to 0} \tau^{-1} < f(\beta x B) >_z$$

with $f(h) = exp(h) - h - 1 > 0$ for $h \neq 0$, is strictly negative:

$$dF/d\tau < 0,$$

and vanishes in the case (by (7.13)

$$dA/d\tau =< dB_\tau >= b(B_\tau) \equiv 0. \tag{7.16}$$

Therefore, the production of free energy as a function of the means A is strictly decreasent whenever

$$b \neq 0,$$

i.e. the trace torsion is not identically zero.

Therefore, we conclude that the strict growth of the entropy ($\frac{-1}{T}F$) is produced by the non-vanishing drift trace-torsion vector field. Furthermore, we have seen that the fluctuation-dissipation equations, which we finally write in the form

$$b(B_\tau) = \kappa(\frac{\partial F}{\partial A}), \quad \text{with} A =< B_\tau >, \tag{7.17}$$

relates the drift vector field evaluated along the stochastic flow equals with its average with respect to the weighted density p_z with $z = \frac{\partial F}{\partial A}$; i.e. one has to rescale the equilibrium measure by the thermodynamic force produced by the variation of the free energy defined by the equilibrium measure along the averaged stochastic flow further coupled to the stochastic flow. Undoubtedly, the fluctuation-dissipation equations not only appear as non-evolutionary, furthermore, they are strictly acausal. The final state of the fluctuations, described by the equilibrium measure is demanded for their formulation, in addition of the knowledge of the integral of the stochastic flow. Thus, entropy growth is **not** related -as currently believed- to the initial conditions of the system. One has changed the equilibrium measure, i.e. the final state of the system by one which incorporates the coupling of stochastic flow to the thermodynamic forces given by the variation of the free energy defined by the equilibrium state with respect to the average of this same flow. In the case of a linear reversible drift vector field, this rescaling is almost tautological, as we shall see in Section VIII.

VIII. DETAILED BALANCE AND THE ONSAGER RECIPROCITY RELATIONS OF LINEAR NON-EQUILIBRIUM STATISTICAL MECHANICS

We have seen in the previous sections, that in the non-linear case, the rescaling of the global structure by introducing local-scale thermodynamic forces, leads to the increase of entropy. This entropy growth is produced by a non-zero drift trace torsion vector field, moreover, one which is of the most general type.

We would like to show in this last section, that as a consequence of the assumption that the drift is **linear** in the space-time coordinates, the rescaling can be obviated since the local thermodynamic forces are represented by the drift itself. In other words, if the global drift is linear, the tendency to equilibrium is represented by the global structure itself; no additional thermodynamic forces are involved. This is a very natural result.

This is precisely the case of the well-known Ornstein-Uhlenbeck process [1,2], also called the harmonic oscillator process; this process has an equilibrium measure introduced by ψ^2 , with ψ a (zero-mean) Gaussian function.

Assume the components $Q_\alpha(x)$ of the trace-torsion one-form are linear functions of x , i.e.

$$Q_\alpha(x) = C_{\alpha\beta}.x_\beta, \quad \text{with} \quad C_{\alpha\beta} = \text{constant}, \tag{8.1}$$

and

$$g_{\alpha\beta}(x) = B_{\alpha\beta} = \text{constant}, \quad \text{for all } \alpha, \beta. \tag{8.2}$$

The Markov process is described by the Ito stochastic differential equation

$$dB(_\tau) = C.B_\tau + D.dW_\tau$$

where $C = (C_{\alpha\beta})$, and D is the matrix with constant entries given by

$$DD^{\dagger} = B.$$

The detailed balance conditions are

$$(\epsilon_\alpha \epsilon_\beta C_{\alpha\beta} + C_{\alpha\beta}) x_\beta = \frac{\partial ln\ p_{eq}(x)}{\partial x_\alpha}, \tag{8.3}$$

and

$$\epsilon_\alpha \epsilon_\beta B_{\alpha\beta} = B_{\alpha\beta}. \tag{8.4}$$

The first condition implies that $p_{eq}(x)$ is a Gaussian function, since $d\ ln\ \psi$ is linear in x, and still, since the first equation does not contain a constant coefficient, then this Gaussian is of zero mean, i.e.

$$\rho = N.p_{eq}(x) = N.\psi^2(x) = N.exp(-\frac{1}{2} x^T \sigma^{-1} x). \tag{8.5}$$

After rearranging and using the symmetry of σ, the stationary Fokker-Planck equation yields

$$-C_{\alpha\alpha} - \frac{1}{2} B_{\alpha\beta} \sigma^{-1}_{\alpha\beta} + (\sigma^{-1}_{\kappa\alpha} C_{\alpha\beta} + \frac{1}{2} \sigma^{-1}_{\kappa\alpha} B_{\alpha\gamma} \sigma^{-1}_{\gamma\beta}) x_\kappa x_\beta = 0. \tag{8.6}$$

Yet, the quadratic term above vanishes if the symmetric part of its coefficient is zero. We may write this condition in matrix form as

$$\sigma^{-1} C + C^T \sigma^{-1} = -\sigma^{-1} B \sigma^{-1} \tag{8.8}$$

or still

$$C\sigma + \sigma C^T = -B. \tag{8.9}$$

The constant term in (8.7) also vanishes if the equation (8.8) is satisfied.

Let us introduce now the matrix $\epsilon = \mathrm{diag}(\epsilon_1, \epsilon_2, \ldots)$; thus,

$$\epsilon^2 = Id.$$

Then, the conditions (8.3, 4) can be written in matrix form as

$$\epsilon C \epsilon + C = -B\sigma^{-1}, \tag{8.10}$$

and

$$\epsilon B \epsilon = B. \tag{8.11}$$

The condition $p_{eq}(x) = N.\psi^2(x)$ is equivalent to the symmetry of σ. (4.8) is

$$\epsilon B \epsilon = B. \tag{8.12}$$

Yet, detailed balance requires that

$$\sigma\epsilon = \epsilon\sigma. \tag{8.13}$$

Multiplying (8.9) by σ yields

$$C\sigma + \sigma C^T = -B, \tag{8.14}$$

which together with (8.10) yields

$$\epsilon C \epsilon \sigma + C\sigma = -B, \tag{8.15}$$

which yield

$$\epsilon C \epsilon \sigma = \sigma C^T, \tag{8.16}$$

which with (8.10) finally yields

$$\epsilon(C\sigma) = (C\sigma)^\dagger \epsilon. \tag{8.17}$$

These are the famous Onsager fluctuation-dissipation of linear non-equilibrium statistical mechanics [4,10]. Note that these relations deal especifically with the linear trace-torsion. The interpretation with respect to a fluctuation-dissipation can be better seen by introducing as a force minus the drift vector field corresponding to the Gaussian function ψ of above

$$F(x) = -grad\ ln\rho = \sigma^{-1}x. \tag{8.18}$$

Here, $F(x)$ is the free energy of the system.

The fluctuation-dissipation relations are the phenomenological equations

$$\frac{d < x >}{d\tau} = \phi(< x >). \tag{8.19}$$

Whenever ϕ is an arbitrary function, then we are in the case of a non-linear fluctuation-dissipation relation, while the particular case of ϕ a linear function yields the linear fluctuation-dissipation relation (8.17).

In the case of the Orstein-Uhlenbeck process,

$$\phi(< x >) \equiv b(< x >) = C. < x > \tag{8.20}$$

which is equal to

$$= C.\sigma F(< x >) = L.F(< x >), \quad \text{where} \quad L = C.\sigma. \tag{8.21}$$

Thus, if the fluctuation-dissipation is such that the **fluxes** $d < x > /d\tau$ are related linearly to the forces $F(< x >) = -1/2 grad\ ln\ \rho$ by a matrix L defined by

$$L = C\sigma, \tag{8.22}$$

i.e.

$$d < x > /d\tau = F(< x >) = L. < x >, \tag{8.23}$$

then the fluctuation-dissipation relations (8.17) yields

$$\epsilon L \epsilon = L^T, \tag{8.24}$$

or

$$L_{\alpha\beta} = L_{\alpha\beta} \quad (\epsilon_\alpha \text{ and } \epsilon_\beta \text{ of the same sign}), \tag{8.25}$$

$$L_{\alpha\beta} = -L_{\beta\alpha} \quad (\epsilon_\alpha \text{ and } \epsilon_\beta \text{ of different sign.} \tag{8.26}$$

These are the Onsager reciprocity relations written in their usual form. Notice also that

$$\epsilon B \epsilon = B, \tag{8.27}$$

and

$$\epsilon \sigma \epsilon = \sigma, \tag{8.28}$$

imply that $B_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ vanish if ϵ_α and ϵ_β have opposite signs.

In the present case of the Ornstein-Uhlenbeck process we have presented, the irreversible components of the drift trace-torsion vector field are zero, so that detailed balance is present. The detailed balance equations give conditions on the coefficients of the fluctuation-dissipation equation, which in the present example are the embodiment of the Onsager reciprocity relations of linear non-equilibrium statistical mechanics.

IX. CONCLUSIONS

Our conclusion that the growth of the entropy follows from a non-zero drift trace-torsion vector field, and that ultimately the existance of a **local** arrow of time is related to it, is new.

Our conclusions stand in contrast with the tenants that stemmed from Prigogine's pioneering work in linear non-equilibrium statistical mechanics, in which the growth of entropy is related to the "internal" fluctuations of the system, which seem to be unrelated to any global (nor geometrical) structure, and unconnected as well to the existance of scales in Nature [25]. The non-linear theory due to Stratonovich [1] stems from similar

hamiltonean notions, yet we have kept his constructions arising from the rescaling of the equilibrium density **in phase space**!, but now in space-time.

Our perspective is completely different to the ones submitted by these authors. In our work, whatever the scale is under examination, the fundamental fact is that the drift trace-torsion vector field carries the geometrical information of the global system, incorporating the fluctuations due to the fact that it is precisely a conjugate (through the metric which describes the fluctuations) vector field, of the trace-torsion differential one-form. Furthermore, at a local "internal" level, the thermodynamic forces produced locally by the system, couple to it through a rescaled metric which is nothing else than the rescaling through the isotopic element ψ of the global metric.

It is an absolutely non-redundant fact that the metric couples to the torsion, even in the case in which the metric reduces to the Euclidean metric $diag(1,\ldots,1)$. The reason for this is that whatever the metric is, there always appears connected with it, as a kind of coupling constant a coefficient -which we purposedly never manifested- multiplying the second order terms ; this coefficient is the diffusion coefficient. It originally appeared in Einstein's theory of Brownian motion; Einstein proved that it is equal to $kT\mu$, where μ is the dissipative term given by the inverse of the friction constant, thus establishing a fluctuation-dissipation relation. In our theory, the non-osmotic dissipative terms are described by the conjugate vector field to the electromagnetic potentials appearing in the de Rham-Kodaira-Hodge decomposition of the trace-torsion, which as we saw, give rise to a conserved probability vector, which historically is related to the non-osmotic dissipative term.

To resume, we have presented a non-linear H-theorem in its free energy version. In a non-covariant setting, this theorem was formulated by Stratonovich [1]. Furthermore, we have related the historical problem of the existance of a "time arrow" with a non-null drift trace-torsion vector field. Yet, what seems to be of extreme relevance in our presentation, is that there is **no** real evolution equation (cf (7.4)) unless one goes from the stochastic flow given by the solution of the Ito stochastic differential equation (3.65), to its mean. Yet, it is a seemingly "miraculous" property of these equations, the fact that the intrinsic time derivative commutes with the operation of taking the mean of the solution flow, and that this commutation settles to the fact that the mean of the Brownian differential is nothing else than the drift trace-torsion vector field, cf. (7.4). Thus, the intrinsic-time evolution is coded into the drift trace-torsion vector field taken along the Brownain solution flow. In other words, there is no time-evolution, and thus no time arrow, unless one has averaged over the fluctuations, passing from the solution of the stochastic flow to its average. Thus, the existance of the time arrow appears to be related with the transition of a continuous non-differentiable stochastic dynamics, to a differentiable dynamics which still is to be evaluated along the stochastic flow. In this sense, one reconciles classical mechanics with a fluctuational dynamics of extreme

complexity. While the Poincaré recurrence theorem for the "classical" (i.e. smooth) dynamical system given by the flow given by integrating the ordinary differential equation

$$dB/d\tau = b(B_\tau), b = grad\ ln\psi,$$

is still valid, our derivation of the non-linear H-theorem relies in the average on the stochastic flow

$$d < B > /d\tau = b(< B_\tau >),$$

with b in its general irreversible form given by (3.46). Thus, the objections by Poincaré to Boltzmann [26,27] are valid if we look on the classical reversible dynamics, i.e. $J \equiv 0$, yet, at the level of the stochastic dynamics, the growth of entropy is valid as long as the drift vector field does not vanish identically. Of course, our conclusions have by-passed completely Boltzmann's constructions and the assumption of irreversibility associated to a "microscopical" hamiltonean. In this work, it is the space-time geometrical structure what determines the local exchanges of thermodynamic fluxes, and ultimately, the existance of a **LOCAL** arrow of time.

A final comment. While our geometries, in the case of the trivial Weyl torsion, are related to the Lie-isotopic geometries, whose field equations we shall present in the second part of this article, the non-trivial components given by the Aharonov-Bohm and Coriolis potentials which give rise to irreversibility, may have a relation with the genotopic structures introduced by Santilli; see Santilli's lectures on these Proceedings.

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THE GEOMETRY OF FLUCTUATIONS, II (QUANTUM GRAVITY AND ERGODICITY)

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Abstract

We complete the geometrization of Relativistic Quantum Mechanics (R.Q.M.) and stochastic quatization of gravitation presented in our first article in these Proceedings, in terms of the Riemann-Cartan-Weyl (RCW) geometries with Weyl-torsion and their associated diffusion processes. The central role is played by the Laplacian operator associated to the RCW geometries. We extend these diffusions on scalars to differential forms, and relate the RCW Laplacian with a deformed Laplacian through the isotopic unit element. The field equations for the RCW geometries are derived from a mean Cartan scalar curvature extremal principle which yields the coincidence between the quantum potential and $1/12R(g)$, $R(g)$ the metric scalar curvature. We introduce the quantum Perron-Frobenius semigroups of ergodic theory and the Lyapunov spectrum, associated to the flows generated by the RCW diffusions. We find that the drift trace-torsion vector field contributes to stabilize the fluctuations. We study topological obstructions to the average stability of these flows which originate in the RCW Laplacian and relate it to the mean growth of entropy.

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INTRODUCTION

In our previous article in these Proceedings [33] (which from now on we shall denote as **I**), we have introduced the Riemann-Cartan-Weyl geometries which give rise to Markov diffusion processes, and shown their fundamental role in describing the relation between a cosmological tendency to equilibrium described by the invariant density $\psi^2 vol_g$, and the local fluctuation-dissipation equations of non-equilibrium, produced by rescaling this density by thermodynamic forces. Locally, the coupling of these forces to the Markov diffusions generated by the cosmological scale geometry, is given by $\psi^2 g$. We gave also the Maxwell-de Rham field equations for the electromagnetic components of the trace-torsion, which break the classical reversibility of the system.

In this second part, we shall show how the diffusions can be built from the field equations; in fact in **I** we gave the field equations for the irreversible components of the trace-torsion vector field, while the field equations for g and ψ were not given. On presenting these field equations we shall show that the relativistic extension of Bohm's quantum potential is $1/12R(g)$, where $R(g)$ is the metric scalar curvature. For carrying this, we shall need to present the relation between the Riemann-Cartan-Weyl laplacian which generates the diffusions, and quantum mechanics as a Hilbert space operator theory. This requires the introduction of Dirichlet regularity conditions, and thus it is treated through the introduction of Dirichlet forms [3,22,38].

We shall also present the description of the classical (i.e. by **smooth curves**) realization of the diffusions, and show that they are deviations of the geodesic flow due precisely to the drift trace-torsion vector field. This shall be encountered through the lagrangian path-integral representation of the heat kernel, which for diffusions in phase space has been obtained by Graham [14] (in formulating statistical mechanics from the point of view of hamiltonean functions), yet which in Graham's work is independent of any symplectic structure and thus it is properly a space-time formalism as proved originally by Takayashi and Watanabe [7]; see [13,25].

In our previous work we have established the relation between the torsion of space-time and the local establishment of a time-arrow. In this second part we shall rather set the structures for an analytic approach to irreversibility, by defining the Perron-Frobenius semigroups associated to the diffusion flows. We shall further introduce the notion of stochastic Lyapunov exponents to express the tendency to equilibrium by entropy growth by showing that the diffusions are unstable. Indeed, the highest Lyapunov exponent expresses the growth of volume elements along the flow which, being unstable, grows with the positive speed described by this Lyapunov number. This global ergodic behavior of the diffusions we shall see that it follows from the topological-geometrical assumption that de Rham first cohomology group of the space-time manifold is not trivial, and such

that it further has an harmonic one-form which preserves the drift trace-torsion vector field b. Thus, the global entropy growth.

All these results stem from a most remarkable fact (whose mathematical proof is most technical, so its proof will be avoided in this article [10]). The stochastic flows given by integrating the diffusions generated by the Riemann-Cartan-Weyl geometries, generate quantum diffeomorphisms of space-time! In account of the non-differentiability of the paths of the diffusion, which are furthermore fractal continuous curves, this points to the stochastic flows generated geometrically, as a most elementary level of dynamics in which the principle of equivalence is verified, in spite at at the level of the classical (i.e. smooth) realizations of the flows, the motion is inherently non-geodesic due to the null drift trace-torsion vector field.

In this article we shall keep the notation of **I**, to which the reader is directed; for the reading of this article, Section III of **I** contains the most relevant geometrical information.

I.C. RCW Geometries and Operator Quantum Mechanics

In this section we shall show that the RCW geometries yield a natural formulation of Quantum Mechanics as an operator theory on the Hilbert space determined by the invariant density of the process: $\psi^2 vol_g$, which we shall assume from now on to be normalized. We shall denote this Hilbert space as $L^2(\psi^2 vol_g)$.

We can associate with the diffusion process a Hamiltonean operator on $L^2(\psi^2 \mathrm{vol}_g)$ [3]. With abuse of notation, let us denote by $H(g,\psi)$ the Friedrichs self-adjoint [38] of the infinitesimal generator $H(g,\psi,0,0)$ (**I**.3.26)

$$H(g,\psi,0,0) = 1/2\triangle_g + g^{\alpha\beta}\partial_\alpha \; ln \; \psi\partial_\beta$$

with domain given by $C_0^\infty(M)$. We can now define the inner product

$$< f_1, f_2 >_\psi = 1/2 \int g(\mathrm{grad} f_1, \mathrm{grad} f_2)\psi^2 \mathrm{vol}_g \tag{1.1}$$

By integration by parts, we obtain

$$< f_1, f_2 >_\psi = -(f_1, H(g,\psi) f_2)_{L^2(\psi^2 vol_g)} \tag{1.2}$$

where $(.,.)_{L^2(\psi^2 vol_g)}$ denotes the inner product in $L^2(\psi^2 \mathrm{vol}_g)$. Let us consider now the closed quadratic form, (the *Dirichlet form*) q associated to $< .,. >$, i.e. $q(f) =< f, f >$ [3,23]. We see from eqt. (1.2) that there is a unique Hamiltonean operator which generates q, it is the self-adjoint operator $-H(g,\psi)$. Since the quadratic form is positive, $q(f) \geq 0$, for any $f \in L^2(\psi^2.vol_g)$, then $H(g,\psi)$ is a negative self-adjoint operator on $L^2(\psi^2.vol_g)$ and the semigroup $exp(\tau H(g,\psi))$ is defined for any $\tau \geq 0$.

Let us see how this construction is related to the usual formulation of Quantum Mechanics in terms of quadratic forms in $L^2(\mathrm{vol}_g)$, which in the non- relativistic flat case has been elaborated by several authors [22]. We consider the mapping $C_\psi : L^2(\psi^2 \mathrm{vol}_g) \to L^2(vol_g)$ defined by multiplication by ψ. This maps takes $C_0^\infty(M)$ into itself. For any f in $C_0^\infty(M)$ we have

$$\begin{aligned} q(\psi^{-1}f) &= <\psi^{-1}f, \psi^{-1}f> \\ &= 1/2 \int \{g(\mathrm{grad}\ f, \mathrm{grad}\ f) - 2g(\mathrm{grad}\ f, \mathrm{grad}\ ln\ \psi)f + g(b,b)f^2\}\mathrm{vol}_g \\ &= 1/2 \int \{g(\mathrm{grad}\ f, \mathrm{grad}\ f) + (\mathrm{div}_g b)f^2 + g(b,b)f^2\}\mathrm{vol}_g \\ &= \int f\{-\frac{1}{2}\triangle_g + V\}f \mathrm{vol}_g = (f, Hf)_{L^2(vol_g)}, \end{aligned}$$

with $b = \mathrm{grad}\ \ln\psi$ and $H = C_\psi \circ H(g,\psi) \circ C_\psi^{-1} = -1/2\triangle_g + V$, where in the weak sense,

$$V = 1/2(\mathrm{div}_g\ b + g(b,b)) = \frac{\triangle_g \psi}{2\psi}, \tag{1.3}$$

is the **relativistic quantum potential**. In the case of $n = 3$ and g the Euclidean metric, we retrieve Bohm's potential in non-relativistic Quantum Mechanics [19].

Thus we have proved that $-H(g,\psi)$ is unitarily equivalent to the Hamiltonean operator $H = -1/2\triangle_g + V$ defined on $L^2(\mathrm{vol}_g)$ and ψ is a generalized groundstate eigenfunction of H with 0 eigenvalue. The non-linear dependence of V on the invariant density introduced by ψ introduces non-local (in the sense of Einstein- Podolsky-Rosen) correlations on the quantum system; we recall that the existance of these correlations have been verified experimentally by Aspect [40]. We shall see below that this dependence of V on ψ is removed due to conformal invariance.

II. The Mean Curvature Extremal Principle

We shall assume that $n = 4$. We start with a general Riemann-Cartan connection (Γ_α^{ab}), (where Greek letters denote space-time indices as until now, and Latin letters denote anholonomic indices), and we introduce its scalar curvature

$$R(\Gamma) = e_a^\alpha e_b^\beta R_{\alpha\beta}^{..ab}, \tag{2.1}$$

where the e_a^α is a field of invertible tetrads with $g_{\alpha\beta} = \eta_{ab} e_\alpha^a e_\beta^b$, with η_{ab} the Euclidean metric, and $R_{\alpha\beta}^{..ab}$ is the curvature tensor of (Γ_α^{ab}) [5]. Definining the following **generalized** conformal (Einstein) Λ transformations [6,4]: i): (Γ_α^{ab}) is invariant, and ii): e_a^α is transformed into $\psi^{-1}e_a^\alpha$; from them we get the usual Weyl transformation on the

metric: $g_{\alpha\beta}$ is transformed into $\psi^2 g_{\alpha\beta}$ while R is transformed into $\psi^{-2}R$. Note that in particular, if $e_a^\alpha = \delta_a^\alpha$, then the Λ transform of the flat Euclidean metric g is the isotopic metric $\psi^2 g$. Since, by above, the scalar fields ϕ transform as $\psi^{-1}\phi$, we get that the functional

$$A(\Gamma, \phi, g) = \int R(\Gamma)\phi^2 \mathrm{vol}_g, \tag{2.2}$$

is conformal invariant. Notice that if ϕ generates a drift vector field $b = \text{grad } ln\phi$, so that $\phi^2 \mathrm{vol}_g$ is the unique invariant density of the corresponding diffusion process, then (2.2) is the **mean Riemann-Cartan scalar curvature**. This theory contains the Einstein-Cartan theory which is obtained by fixing the ϕ to be positive constant [5], and consequently it contains the classical Einstein's theory, retrieved by symmetrizing Γ. Taking variations with respect to g we obtain that

$$R_{\alpha\beta}(\Gamma) - 1/2 g_{\alpha\beta} R(\Gamma) = 0, \tag{2.3}$$

i.e. the Einstein-Cartan equations for Γ in the vacuum, while by taking variations with respect to $T^\gamma_{\alpha\beta}$, we obtain that

$$T^\gamma_{\alpha\beta} = \delta^\alpha_\beta \ \partial\gamma \ ln\phi - \ \delta^\alpha_\gamma \partial_\beta \ ln\phi, \tag{2.4}$$

so that, up to normalization, $Q = d \ ln\phi$. Hence, we have obtained the exact component of the trace-torsion one form. Taking variations with respect to ϕ we get the teleparallelism: $R(\Gamma) = 0$; replacing (2.4) in (2.3) we get the field equations

$$G_{\alpha\beta}(g) = -\frac{6}{\phi^2} T_{\alpha\beta}, \tag{2.5}$$

with $G_{\alpha\beta}(g)$ the Einstein **metric** tensor, and

$$T_{\alpha\beta} = \partial_\alpha\phi \ \partial_\beta\phi - 1/2 g_{\alpha\beta}\partial_\gamma\phi\partial^\gamma\phi - \frac{1}{6}(\nabla_\alpha\nabla_\beta\phi^2 - g_{\alpha\beta}\triangle_g\phi^2), \tag{2.6}$$

minus the improved energy-momentum density of the renormalizable gauge theories [20]. Note that our theory extends Einstein's relativity in that it has a non-trivial source, which furthermore is of geometrical origin: it appears from the trace torsion. Now, by taking the trace in (2.5) we finally get

$$(\triangle_g - \frac{1}{6}R(g))\phi = 0, \tag{2.7}$$

so that ϕ is a generalized groundstate of the conformal invariant wave operator defined on $L^2(\mathrm{vol}_g)$. Note that from (2.7) we conclude that the quantum potential is $\frac{1}{12}R(g)$ which **does not** depend on the scalar field ϕ at all. Therefore, the correlations on the

quantum system are mediated by the metric scalar curvature! We furthermore note that this identity restores the validity of the superposition principle, otherwise violated by (1.3).

Probably this profound result should not surprise us. The non-local phenomenology of local geometric origin given by the quantum potential is another manifestation of non-local phenomenae related to (Yang-Mills) curvature, as the Aharonov-Bohm effect and Berry's phase [41]. Yet, it is still remarkable that the (sign of the) metric scalar curvature plays a central role in the existance of non-trivial solutions of Witten's monopole equations [42].

Solving the conformal invariant wave equation for the field (we change notation to the original) ψ with Dirichlet regularity conditions we obtain a conformally conjugate Dirichlet form whose associated amiltonean operator is $-H(g,\psi)$, and thus the Markovian semigroup determined by it can be reconstructed. Let us show this in detail.

Suppose that there exists a positive compact supported C^2 function ψ on M such that

$$H\psi := (\triangle_g - \tilde{V})\psi = 0,$$

where $\tilde{V}$ denotes the operator of multiplication by $KR(g)$, $K = \frac{1}{6}$. Put $X = \frac{\triangle_g \psi}{\psi}$; then, X is a continuous function. Consider the quadratic form on $C_0^1(M)$ given by

$$Q(f) = \frac{1}{2}\int (|df|^2 + \tilde{V}f^2) vol_g.$$

Put $h = \psi^{-1}f$, then

$$Q(f) = \frac{1}{2}\int (|d(\psi h)|^2)^2 + \tilde{V}h^2\psi^2) vol_g$$

$$= \frac{1}{2}\int (|dh|^2\psi^2 + h^2|d\psi|^2 + 2 < dh, d\psi > h\psi + \tilde{V}h^2\psi^2) vol_g$$

$$= \frac{1}{2}\int (< d\psi, d(\psi h^2) > + |dh|^2\psi^2 + \tilde{V}h^2\psi^2) vol_g$$

which, by integration by parts of the first term, gives

$$= \frac{1}{2}\int (|dh|^2 + (\tilde{V} - X)h^2)\psi^2 vol_g$$

$$= \frac{1}{2}\int |dh|^2\psi^2 vol_g \geq 0.$$

We have proved that Q is positive. Then, the form $Q_\psi(f) := Q(C_\psi f) = Q(\psi f)$, defined on $f \in L^2(\psi^2 vol_g)$, equals to

$$\frac{1}{2}\int |dh|^2\psi^2 vol_g$$

is positive, and its Hamiltonean operator, $-H(g,\psi)$ is negative self-adjoint on $L^2(\psi^2 vol_g)$; consequently, $exp(\tau H(g,\psi))$ is a Markovian semigroup on $L^2(\psi^2 vol_g)$, or in its dense domain $C_0^\infty(M)$.

Therefore, starting from a positive compact supported solution of the equation (2.7), we have reconstructed the Markovian semigroup generated by the RCW Laplacian $H(g,\psi)$.

We shall now establish the relation between the heat kernel $p_{\rm conf}(\tau,x,y)$ of the Markovian semigroup $exp(\frac{\tau}{2}H)$ and the heat kernel $p^\psi(\tau,x,y)$ of the semigroup generated by $H(g,\psi)$. We have

$$exp(\tau H(g,\psi))f(x) = \psi^{-1}(x)exp(\frac{\tau}{2}H)(\psi f)(x) = \int \psi^{-1}(x)p_{\rm conf}(\tau,x,y)\psi(y)f(y)vol_g(y)$$

so that we conclude that

$$p^\psi(\tau,x,y) = \psi^{-1}(x)\psi(y)p_{\rm conf}(\tau,x,y).$$

Thus, we have linked the quantization in the two Hilbert spaces, the groundstate Hilbert space $L^2(\psi^2 vol_g)$, and $L^2(vol_g)$. The former corresponds to the RCW geometry, while the latter is the usual Hilbert space for quantization of the scalar field ψ in terms of the Riemannian invariants of the manifold M described in terms of g. We remark that the introduction of both spaces and the unitary transformation between them, has allowed us to identify the quantum potential as the scalar metric curvature affected by a coefficient which enforces the conformal invariance, while working only in the usual Hilbert space would not have allowed for this identification. Thus, in the $L^2(vol_g)$ space we have the Hamiltonean operator obtained by B.de Witt in his pioneering work [9], and reencountered by several researchers in quantum field theory in Riemannian geometries, which consider the HamiDew expansion of $p_{\rm conf}(\tau,x,x)$ [37]. Yet, our result is in disagreement with the path integral representation in a Riemann-Cartan geometry in this **same** Hilbert space due to Kleinert, in which he obtains $2\tilde{V}$ for "effective" potential added to the Laplace-Beltrami operator (see, chap. X, [34]).

We want finally to construct the Markovian semigroup associated to the complete Laplacian $H(g,\psi,A_1,A_2)$ of our previous article, where

$$A_1 = \delta\beta_2/\psi^2,$$

is a co-closed one-form with β_2 a smooth two-form on M, and the harmonic one-form

$$A_2 = \omega_1/\psi^2,$$

with ω_1 an harmonic one-form on M. To do this, one has simply to solve additionally the de Rham-Maxwell field equations presented in **I**; cf. [32] for a thorough treatment of the non-general applicability of the minimal coupling rule.

This concludes the construction of the diffusions generated by $H(g,\psi,A_1,A_2)$.

III. The Classical Most Probable Realizations of the Quantum Motions

Let us determine classical gravitational motions **from** the quantum motions $\{B(\tau) : \tau \geq 0\}$ defined by integrating eqts, (**I**.3.65) [4].

Let $\{P_x\,,\ x \in M\}$ be the Markovian system on M determined by $\{P_\tau, \tau \geq 0\}$, with infinitesimal generator given by $H(g,\psi,A_1,A_2)$ of (**I**3.26,3.33,3.34):

$$P_x[B(\tau) \in \mathcal{B}\} = p^\nabla(\tau, x, \mathcal{B}),$$

, with $\mathcal{B}$ a Borel measurable set of M. We recall from **I** that $p^\nabla(\tau,x,\mathcal{B})$ is the transition density of the most general Markovian semigroup generated by $H(g,\psi,A_1,A_2)$.

Take an arbitrary **smooth** curve on M, $\varphi : [0,\infty) \to M$, $\varphi(0) = x$, and consider the "tube" of radius ε centered on φ, $\varepsilon > 0$, of all quantum motions $B(\tau)$ starting at x, $\tau \in [0,T]$, $T > 0$,

$$T_\varepsilon(\varphi) = \left\{ B(\tau) : B(0) = x,\ \sup_{\tau\in[0,T]} \|B(\tau) - \varphi(\tau)\| \ < \varepsilon \right\}\ , \tag{3.1}$$

where $\|\ \|$ denotes the Riemannian distance. We are interested in the asymptotic expression of $P_x[T_\varepsilon(\varphi)]$ as $\varepsilon \to 0^+$. This expression is derived in the theory of large deviations in probability theory [13] and has been the matter of research in the semiclassical approach to quantum gravity [16].

We have

$$P_x[T_\varepsilon(\varphi)] \approx e^{-\lambda_1 T/\varepsilon^2}\ e^{-\int_0^T 1/2L(\varphi,\dot\varphi)(s)\ ds}\ , \tag{3.2}$$

where L is the so-called Onsager-Machlup lagrangian or probability functional [7,13]

$$L(\varphi,\dot\varphi) = \|b - \dot\varphi\|^2 + \mathrm{div}_g b - 1/6R(g)\ . \tag{3.3}$$

(The clearest proof of this result, as given in [7] is extremely involved and will not be presented here; a simpler yet unpublished proof has been obtained recently by Watanabe).

Note that L contains the whole information on the RCW geometrical structure, i.e. on g and b.

The field equations for

$$b = grad\ ln\ \psi + 1/\psi^2(\delta\beta_2 + \omega_1), \tag{3.4}$$

are given by eqts. (2.7) and the de Rham-Maxwell equations presented in **I**. Since $(1/\psi^2)\delta\beta_2$ is co-closed, by construction, while ω_1/ψ^2 is harmonic, the divergence of their conjugate vector fields vanish, and consequently we have the reduced term in the expression of L:

$$div_g b = div_g\ grad\ ln\ \psi.$$

Yet, taking in account that the quantum potential

$$1/2(div\ grad\ ln\ \psi + ||grad\ ln\ \psi||^2) = 1/2(div_g b + ||grad\ ln\ \psi||^2)$$

equals $1/12R(g)$, we conclude that L takes the reduced modified form

$$1/2L(\varphi, \dot{\varphi}) = 1/2\ ||\dot{\varphi}||^2 - < \dot{\varphi}, b > \ . \tag{3.5}$$

Thus, the Onsager-Machlup lagrangian is nothing else that the modified kinetic energy of φ due to the term

$$- < \dot{\varphi}, b >$$

which is nothing else than (minus) the projection of $\dot{\varphi}$ along the **full** drift trace-torsion vector field $grad\ ln\ \psi + 1/\psi^2(\delta\beta_2 + \omega_1)$.

To obtain the classical realizations, φ, of $\{B(\tau) : \tau \geq 0\}$, we extremize (3.5), so that they are the **most probable** realizations of the quantum motions. Note that whenever b is orthogonal to the classical velocity vector field $\dot{\varphi}$ we obtain the geodesic flow of g as the most probable approximation. Otherwise, in the particular case of a local gauge in which ψ is positive constant, and we further assume that we have the detailed balance condition $J \equiv 0$, then $b = 0$, and the most probable approximation is the geodesic flow of g.

This requires further discussion. It is well known that relativistic neutral **spinless test**-particles are uninfluenced by the torsion of the background Cartan geometry [2],[8] and follow the geodesic flow determined by the metric of the Cartan connection. In our above derivation of the classical realizations of the stochastic motions, we see that in contrast with the classical test-particle situation just described, the realization of the quantum motions deviate from the geodesic flow due to $b = \text{grad ln}\psi$. As shown above, this is due to the fact that from the conformal invariance of the field equations, the **relativistic quantum potential**, we have that $1/2\left(||grad\ ln\ \psi||^2 + \text{div}_g\ b\right) = \triangle_g\psi/2\psi$, equals to $1/12R(g)$, which no longer depends on the density at all. Yet, it should be remarked that these classical realizations defined by the lagrangian (3.5) do not conflict with the principle of equivalence, since they are **not** test-particles. We would like to remark still that our geometrical characterization of the quantum potential was out of the scope of Bohm and Vigier's proposal for non-relativistic Q.M. [19]-, shedding new light on the quantum correlations produced by the quantum potential [24]. The identification of the quantum potential with $\frac{1}{12}R(g)$ is thus the core of the quantization of gravitation in the present theory.

We recall now the basic structures of the ergodic theory of dynamical systems [17]. We consider a dynamical system $\Theta_{\tau;\tau\geq 0}$ defined on a topological space S, $\Theta_\tau : S \to S$, for any $\tau \geq 0$ verifying the semigroup composition property: $\Theta_{\tau+\tau'} = \Theta_\tau \circ \Theta'_\tau$.

A fundamental example is provided by the solution flows of ordinary differential equations on S a smooth manifold, an instance of which is the geodesic flow of a metric on S. In these cases, $\Theta_\tau(x)$ is the position at time τ of the integral curve of the ordinary differential equation, where the initial value has been fixed.

To set the basis of an ergodic theory we need more structure on S, namely a σ algebra of sets $\mathcal{F}$ on S such that the inverse image by Θ_τ of an arbitrary measurable set $F \in \mathcal{F}$ is a measurable set in F. We further require the existance of a probability measure μ on S, i.e.: $\mu(S) = 1$, which is further invariant by the flow,i.e.: i.e.

$$\mu(\Theta_\tau^{-1}(F)) = \mu(F), \tag{4.4}$$

for any $F \in \mathcal{F}$, and $\tau \geq 0$. The role of this invariant measure is that of an **equilibrium** measure on S.

Thus, we consider the triple $(S, \mathcal{F}, \mu)$, which in the language of the theory of dynamical systems is called the ***phase space*** (the terminology is a reminiscence of classical mechanics) of the dynamical system $\{\Theta_\tau\}$. Then one introduces the following semigroups. Firstly, let $f \in L^\infty(S)$; we define the Koopman operators:

$$(V_\tau f)(y) = f(\Theta_\tau(y)), \tag{4.5}$$

for any $\tau \geq 0$; then we have the semigroup property:

$$V_{\tau+\tau'} = V_\tau \circ V_{\tau'} \tag{4.6}$$

on $L^\infty(S)$. We finally introduce the Perron-Frobenius operators,

$$U_\tau, \tau \geq 0$$

defined as follows: For any r a density on S and $F \in \mathcal{F}$,

$$\int_F (U_\tau r)(y) d\mu(y) := \int_{\Theta_\tau^{-1}(F)} r(y) d\mu(y), \tag{4.7}$$

which is straightforward to see that if also defines a semigroup: the *Perron-Frobenius (PF)* semigroup of operators. With respect to the pairing on Y defined by the measure μ, these semigroups are adjoint, i.e.

$$U_\tau^\dagger = V_\tau,$$

for any $\tau \geq 0$.

These are the basic analytical structures in terms of which the ergodic theory of dynamical systems is formulated [17;18a].

For diffusions processes, in which quantum motions are described by equations (**I**.3.65) we can introduce similar structures in terms of the stochastic flow defined by integrating these equations. Yet, remarkably we obtain that for these flows, the usual composition rules are unvalid. We have instead:

$$F_{\tau+\tau'}(\omega) \;=\; F_{\tau}(\theta_{\tau'}(\omega)) \circ F_{\tau'}(\omega), \tau, \tau' \geq 0, \tag{4.8}$$

almost surely for $\omega \in \Omega$, with

$$\theta_{\tau} : \Omega \to \Omega$$

the canonical shift on Wiener space: for any

$$s \mapsto \omega(s),$$

the canonical shift is the mapping

$$\theta_{\tau}(\omega)(s) = \omega(\tau + s) - \omega(\tau).$$

Here, Ω is the canonical Wiener space of maps $\omega : R \to R^n, \omega(0) = 0$, with the canonical realization of the Wiener process $W_{\tau}(\omega) = \omega(\tau)$.

Consider the enlarged space $S = M \times \Omega$, and the mapping

$$\Theta_{\tau} : S \to S, \Theta_{\tau}(x, \omega) := (F_{\tau}(\omega)(x), \theta_{\tau}(\omega)), (x, \omega) \in M \times \Omega. \tag{4.9}$$

Then, since

$$F_{\tau}(F_{\tau'}(x, \omega), \theta_{\tau'}(\omega)) = F_{\tau+\tau'}(x, \omega), a.s., \tag{4.10}$$

we have,

$$\Theta_{\tau+\tau'} \;=\; \Theta_{\tau} \circ \Theta_{\tau'}, \tau, \tau' \geq 0, \text{a.s.}. \tag{4.11}$$

Furthermore, if ρ is an invariant measure for the Markovian semigroup associated to the s.d.e., then the product measure

$$\mu = \rho \otimes P, \tag{4.12}$$

is **invariant** by the flow:

$$\mu(\Theta_{\tau}^{-1}(B \times \Lambda)) = \mu(B \times \Lambda), \tag{4.13}$$

for any Borel measurable sets $B \in \mathcal{B}(M), \Lambda \in \mathcal{B}(\Omega)$; indeed, for any such B and Λ we have:

$$\begin{aligned}\rho \otimes P(\Theta_\tau^{-1}(B \times \Lambda)) &= \rho \otimes P\{(x,\omega) : F_\tau(\omega)x \in B,\ \theta_\tau(\omega) \in B\} \\ &= \rho \otimes P\{(x,\omega) : F_\tau(\omega)x \in B\}.\rho \otimes P\{(x,\omega) : \theta_\tau(\omega) \in \Lambda | F_\tau(\omega)x \in B\} \\ &= (\int_{x \in M} p_\tau(x,B)d\rho(x)).\rho \times P\{(x,\omega) : \theta_\tau(\omega) \in \Lambda\} \\ &= \rho(B).P(\Lambda)\end{aligned} \tag{4.13}$$

since ρ is invariant and the events of the conditional probability are independent.

Consider the triple $(S, \mathcal{F}, \mu)$, where $\mathcal{F}$ is the σ-algebra $\mathcal{B}(M) \times \mathcal{B}(\Omega)$, the product of σ algebras of measurable sets on M and Ω respectively,

$$S = M \times \Omega$$

and

$$\mu = \rho \times P.$$

Then, $(S, \mathcal{F}, \mu)$ is a *stochastic phase space* with μ a Θ_τ-invariant measure representing an equilibrium measure. We introduce the *stochastic Koopman semigroup* of operators: $(V_\tau f)(y) = f(\Theta_\tau(y)), y \in S, f \in L^\infty(S)$ and a *stochastic PF semigroup* defined on the probability densities r on S, by (4.5), for our present stochastic phase space. Then, $V_\tau^\dagger = U_\tau, \tau \geq 0$.

In the case of R.Q.M. as constructed above, for a compact orientable space-time manifold, M, and $H(g, \psi, A_1, A_2)$ is an elliptic operator with $\rho(B) = \int_B \psi^2 \mathrm{vol}_g$, $B \in \mathcal{B}(M)$, the unique invariant measure of the quantum flow defined by integrating eqts. (I.3.65) with b given by (3.4), we have a *quantum-stochastic PF semigroup* determined by this flow. Therefore, from the fact that stochastic flows have the diffeomorphic property alike classical smooth flows, we have a **stochastic** point dynamics on M **enlarged** by the canonical shift of Ω. Yet, in this setting one is interested in the M-part of the stochastic flow while its Ω-part is considered to be inaccesible [11]. Thus, we see that the construction of the stochastic extensions of the probabilistic structures of ergodic theory, require in addition of the equilibrium measure provided by the time-invariant solution of the Fokker-Planck equation, the Wiener standard measure. Therefore, the present ergodic theory requires the inclusion of quantum field theory, described by the standard Gaussian measure.

We shall see in the following Section, that we have a complete setup to persue the construction of an ergodic theory of quantum gravity. We remark here the simplicity of the invariant measure for the ergodicity studies of the flows associated with the RCW geometries: **the product of the Born measure and Wiener's measure**. While at

the level of stochastic paths the processes have measures which depart from the Gaussian measure due precisely to the torsion, at equilibrium the interactions are taken in account by the Born measure, while the measure on paths continues to be Gaussian. This can be formalized further in lifting (**I**.3.65) to a s.d.e. valued in vector fields, and as such we consider the associated measure on **dynamical systems**. For this lifted s.d.e., the drift is now **constant** and equal to the given drift torsion vector field, and thus the measure on vector fields is a **Wiener** measure affected by a **constant**. Thus, the predominance of the Wiener measure for dealing with the interaction theory we have presented, appears to be mandatory. This is quite remarkable in view of the well known failure of attempts to construct non-trivial interaction models in quantum field theory. Once the drift trace-torsion vector field is fixed, the quantum field theory is as if the interactions would be trivial.

V. THE LYAPUNOV SPECTRUM AND THE ψ-DEFORMED LAPLACIAN

VI.A. The stochastic Lyapunov spectrum

Having presented an invariant measure for the flows in the augmented state space, we shall proceed now to introduce the Lyapunov spectrum and study the stability perfile of these flows. Finally, we shall relate this perfile, with a deformed laplacian which we shall prove to be conformally conjugate to the RCW laplacian, in a way such that the non-stability topological conditions on the flows are described by the eigenvalues of these operators.

For any $h \in L^2(M, \rho)$ and for $n \in N$, we put

$$P^n f(x) = \int p^{\nabla}(n, x, dy) f(y)$$

where $p^{\nabla}(\tau, x, y), \tau > 0$, is the transition density of a stochastic flow on M generated by $H(g, \psi, A_1, A - 2)$ with invariant density given by ρ. We introduce further the Cesaro sums

$$f^n(x) = \frac{1}{N} \sum_{i=0}^{N-1} P^n f(x).$$

By von Neumann's Ergodic Theorem, $f_n \to f^*, n \to \infty$, where the limit is taken in the $L^2(M, \rho)$ sense. For $f \in L^1(M)$, define $P^n f$ and f^n as above. Then, by Birkhoff's Ergodic Theorem we have: $f^n(x) \to f^*(x), n \to \infty$, for ρ almost everywhere $x \in M$. As usual, $f^*(x)$ is the (internal) time average of f [36].

The Markov stochastic system with transition density $p_\tau(x,y), x, y \in M$ is called ergodic with respect to its invariant measure ρ, if for any $f \in L^1(M,\rho)$, $f^*(x)$ is ρ-constant a.e.. Consequently, $f^*(x) = \hat{f}(x)$, with $\hat{f}(x) = \int f(x)d\rho(x) = \int f(x)\psi^2 vol_g$, a.e..

Let us assume as above, that we have a Markov system written in the form of the Ito s.d.e. (**I**.3.65). We already know that with appropiate regularity conditions on Y and Q, eqt.(17) has an integral flow $F_\tau(\omega) : M \to M$, for a.e. ω, and any $\tau > 0$, which is a diffeomorphism of M, with unique invariant density ψ^2, and on the augmented space $M \times \Omega$ we have the unique invariant measure $\rho \otimes P$, $\rho(B) = \int_B \psi^2(x) vol_g(x)$, $B \in \mathcal{B}(M)$. It was proved by Carverhill ([10.*b*]) that there exists a sub-set $\Gamma \subset M \times \Omega$ of full measure (i.e.: $\rho \otimes P(\Gamma) = 1$), such that for $(x,\omega) \in \Gamma$, there exists a filtration of $T_x M$ by linear sub-spaces:

$$0 = V^{(r)}_{(x,\omega)} \subset V^{(r-1)}_{(x,\omega)} \subset \ldots \subset V^{(1)}_{(x,\omega)} = T_x M,$$

together with real numbers

$$\lambda^{(r)}_{(x,\omega)} < \lambda^{(r-1)}_{(x,\omega)} < \ldots < \lambda^{(1)}_{(x,\omega)},$$

such that, for each $j \in \{1,\ldots,r\}$, we have:

$$v \in V^{(j)}_{(x,\omega)} - V^{(j+1)}_{(x,\omega)}$$

if and only if

$$lim_{t\to\infty} \frac{1}{t} ln \|T_x F_\tau(\omega) v\| = \lambda^{(j)}_{(x,\omega)}.$$

The numbers $\{\lambda^j_{(x,\omega)}, j = 1,\ldots,r\}$ are the ***characteristic*** or ***Lyapunov exponents*** of the system (17) with respect to ρ. When the system is ergodic with respect to ρ, the exponents are $\rho \otimes P$ almost surely independent of (x,ω); in this case we shall simply denote them as $\{\lambda^j, j = 1,\ldots,r\}$. We shall still assume that M is compact, then the choice of the Riemannian norm $\|.\|$ in the definition is irrelevant, as well as the choice of the metric on the definition of the stable manifold below.

Thus, we have presented a stochastic system version of Ruelle's construction for classical dynamical systems. We have also a stochastic version of a stable manifold.

If $\lambda^j < 0$, then for $(x,\omega) \in \Gamma$ as above, the subset

$$\mathcal{V}^j_{(x,\omega)} = \{y \in M; \overline{\lim}_{t\to\infty} \frac{1}{t} ln\ d(F_\tau(\omega)(y), F_\tau(\omega)(x)) \leq \lambda^j\},$$

is called the ***stable manifold***.

We shall give, without proof, a formula due to Carverhill [10] for the explicit computation of the Lyapunov spectrum.

Let T^0M denote the non-null tangent bundle to M given by removal of the zero-section of TM, and SM is the unit sphere tangent bundle: $\{v \in TM : ||v|| = 1\}$. We consider the radial projection mapping $G : T^0M \to SM$. Take any $v \in SM$. We then have a stochastic process $GTF_\tau(\omega)v$ in SM, which we denote as $\eta_\tau(\omega)v$. It is proved (see Lemma 3.3, [10b]) that $\eta_\tau(\omega)v$ is the solution of the stochastic process $\hat{X}$ in SM with $\hat{X}$ the vector field on SM given by $\hat{X}(v) = TG.\nabla X(x)v, v \in SM$. We remark here the reintroduction of the Levi-Civita covariant derivative associated to the Riemannian metric g, which we denote as before as ∇. Take any $x \in M$ such that the Lyapunov spectrum at (x, ω) exists for a.e. ω. (This is so, for ρ a.e. x.) Take any $v \in S_xM$. Then the Lyapunov number $\lambda_{(x,\omega)}$ is given a.s. by

$$\lambda_{(x,\omega)} = lim_{t\to\infty} \int_0^t (g_b(\eta_s(\omega)v) + \frac{1}{2}\sum_{i=1}^{n} dg_Y(\hat{Y}(\eta_s(\omega)v)e_i, e_i))ds. \qquad (5.1).$$

Here,

$$g_b : SM \to R, g_b(v) =< v, \nabla b(x)v >, v \in SM,$$

$$g_Y : SM \times R^n \to R^n, g_Y(v, e) =< v, (\nabla Y(x)v)e >, v \in SM, e \in R^n,$$

and $\hat{Y}(v) = TG.\nabla Y(x)v$. Here, we recall from **I**, b, is the drift trace-torsion vector field described in (3.4).

Now,

$$g_b(v) = - < v, div\ grad\ ln\psi.v >= \triangle_g ln\ \psi, \qquad (5.2)$$

for any $v \in SM$ (here we have used the fact that both $(1/\psi^2)\beta_2$ and $(1/\psi^2)\omega_1$ are both divergenceless (de Rham-Kodaira-Hodge decomposition introduced in **I**) ; then, the first term in (5.1) is

$$lim_{t\to\infty} \frac{1}{t} \int_0^t \triangle_g ln\ \psi(\eta_s(\omega))ds$$

which, if we assume ergodicity with respect to ρ, i.e. the time average equals the ρ-average, and thus we get

$$\int (\triangle_g ln\psi)\psi^2 vol_{(g)}$$

which by integration by parts yields

$$-2\int ||\mathrm{grad}\psi||^2 vol_g(x) = -2\int ||\mathrm{grad}\ \ln\psi||^2 \rho(x) vol_{(g)}.$$

Therefore, the contribution of the drift to the Lyapunov exponent is, up to a constant, negative and given by minus the average kinetic energy of the drift vector

field. Thus, the **drift trace – torsion vectorfield contributes to stabilize** the system. Yet, we still have the contribution due to the noise term present in the second term of (5.1). Before we actually face the problem of determining the sign of the Lyapunov exponent computed by formula (5.1) we remark that the computability of the Lyapunov exponents through formula (5.1) is essentially limited to the highest exponent, $\lambda^{(1)}$.

Indeed. Let $\Gamma \subset M \times \Omega$ be a set of full $\rho \otimes P$ measure on which the Lyapunov system can be defined. Let us assume that the system is ergodic with respect to ρ. Let us denote still by Γ a full measure subset in $M \times \Omega$ such that the Lyapunov system is defined and independent of (x,ω). For each $x \in M$, put $\Gamma_x = \{\omega : (x,\omega) \in \Gamma\}$. Then, $P(\Gamma_x) = 1$, for ρ a.e. x. At such an x, the Lyapunov spectrum is not random, yet the filtration itself may be random (in any case, the filtration is measurable). In this case, we have for certain $j \in \{1,\ldots,r\}$, that

$$P\{\omega : lim_{t\to\infty}\frac{1}{t} ln\ ||T_x F_\tau(\omega)v|| = \lambda^{(i)} = 0\},$$

for all $v \in S_x M$. In this case, the formula (33) will not see this element of the Lyapunov spectrum. Which exponents are then computed by formula (23)?. With the assumptions of above, we have that for Lebesgue a.e. $v \in S_x M$,

$$lim_{t\to\infty}\frac{1}{t} ln||TF_\tau(\omega)v|| = \lambda^{(1)},$$

for a.e. ω (see [10.*b*]). Thus, starting from a.e. v, formula (33) only computes the highest Lyapunov number, $\lambda^{(1)}$.

We shall say that the flow is stable, if the highest Lyapunov exponent is negative.

For analyzing the stability of the quantum flows, we shall introduce a stronger notion of stability called moment stability.

VI.B. THE ψ-DEFORMED LAPLACIAN AND THE RCW STOCHASTIC FLOWS

We repeat here some notions presented in Section III of **I**. Let us assume in the following that we have a smooth n-dimensional orientable compact manifold M provided with a Riemannian metric, g. We consider the Hilbert space of square summable ω of differential forms of degree q on M, with respect to vol_g. We shall denote this space as $L^{2,q}$ or still as $L^2\Omega^q(M,\mathrm{vol}_g)$. The inner product is

$$< \omega, \phi >= \int_M < \omega_x, \phi_x > \mathrm{vol}(g) = \int_M \omega(x)_{\alpha_1\ldots\alpha_q}\phi(x)^{\alpha_1\ldots\alpha_q}\mathrm{vol}_g \qquad (6.1)$$

where $\phi(x)^{\alpha_1 \cdots \alpha_q}$ is the covector at $x \in M$ conjugate to the q-covector $\phi(x)$, for $\phi, \omega \in L^{2,q}$.

The de Rham-Kodaira operator on $L^{2,q}$ is defined as

$$\triangle = -(d+\delta)^2 = -(d\delta + \delta d),$$

where δ is the formal adjoint defined on $L^{2,q+1}$ of the exterior differential operator d defined on $L^{2,q}$:

$$< \delta\phi, \omega > = < \phi, d\omega >,$$

for $\phi \in L^{2,q+1}$ and $\omega \in L^{2,q}$. In the case of $q = 0$, this is the Laplace-Beltrami operator on functions encountered before; in the general case we have in addition of $tr\nabla^2$ the contribution of the Weitzenbock curvature term. Let us assume that the real valued function on M ψ is smooth and everywhere positive. We then have an induced smoooth density $\rho = \psi^2 \mathrm{vol}_g$ on M.

We now introduce the Hilbert space $L^{2,q,\rho} = L^2\Omega^q(M,\rho)$, of differential forms on M of degree q, square integrable with respect to ρ, with inner product:

$$< \phi_1, \phi_2 >^\rho = \int_M < \phi_1(x), \phi_2(x) > \rho, \tag{6.2}$$

for $\phi_1, \phi_2 \in L^{2,q,\rho}$. On a space-time manifold with a Riemannian metric, this inner product is covariant, and it clearly contrasts in this respect with the usual inner product which demands the choice of a constant time hypersurface. We define the quadratic form $q(\phi) = \frac{1}{2} < \phi, \phi >^\rho$, with ϕ on the Hilbert space given by the completion of the space of all smooth q-forms under the $L^{2,\rho}$ inner product. In the case of exact one-forms, this is quadratic form introduced in correspondance with the Brownian processes determined by $H(g,\psi)$. The Markovian semigroups we are interested are associated to the closed extension of q with Hamiltoneans (or what is the same, with infinitesimal generators [23]) which will be given by the Friedrichs self-adjoint extensions of the laplacian operators we shall compute below.

Consider the formal adjoint of d, which we shall denote as δ^ψ defined on $L^{2,q+1,\rho}$ as follows

$$< \delta^\psi \omega, \phi >^\rho = < \omega, d\phi >^\rho, \tag{6.3}$$

for any $\omega \in L^{2,q,\rho}$ and $\phi \in L^{2,q,\rho}$. Since $d^2 = 0$, we have

$$(\delta^\psi)^2 = 0. \tag{6.4}$$

For any smooth function f defined on M, and ω a q-form:

$$\delta(f\omega) = f\delta\omega - i_{\mathrm{grad}\ f}\omega,$$

where i_X is the interior product derivation on q-forms.

We introduce the operator on $L^{2,q,\rho}$:

$$\triangle^{\psi,q} = -(d + \delta^{\psi})^2, \tag{6.5}$$

which still writes as

$$-(d\delta^{\psi} + \delta^{\psi} d).$$

Recalling the definition of the Lie-derivative operator $L_X = di_X + i_X d$, X a smooth vector field on M, we finally have

$$\triangle^{\psi,q} = \triangle^{q} + 2L_{\text{grad } \ln\psi}, \tag{6.6}$$

Let us define now the deformed exterior differential operator mapping q-forms in $q+1$-forms, by:

$$d^{\psi} = \psi d \psi^{-1}, \tag{6.7}$$

so that

$$d^{\psi}\omega = d\omega - d\ln\psi \wedge \omega.$$

We have that

$$(d^{\psi})^2 = 0. \tag{6.8}$$

We introduce now the deformed co-differential operator as the formal adjoint of d^{ψ}:

$$(d^{\psi})^* = \psi^{-1}\delta\psi. \tag{6.9}$$

We introduce the deformed Laplacian operator defined as:

$$L^{\psi,q} = -(d^{\psi} + d^{\psi *})^2, \tag{6.10}$$

which can still be written as

$$-(d^{\psi} d^{\psi *} + d^{\psi *} d).$$

We have the following relation between the two Laplacian operators:

$$\triangle^{\psi,q} = \psi^{-1} L^{\psi,q} \psi, \tag{6.11}$$

so that these two operators are conformally equivalent under conjugation by ψ. Note that $\triangle^{\psi,0} = 2H(g,\psi)$.

From the fact that $(d^{\psi})^2 = 0$, we can define a deformed de Rham complex: $H^q_{\psi}(M,R)$ by $Ker(d^{\psi} : \Lambda^q \to \Lambda^{q+1})/Ran(d^{\psi} : \Lambda^{q-1} \to \Lambda^q)$. Yet, since $Ker(d^{\psi}) = \psi Ker(d)$, and $Ran(d^{\psi}) = \psi Ran(d)$, we obtain that $H^q_{\psi}(M,R) \cong H^q(M,R)$, for any $q = 0, \ldots, n$.

Now, by Hodge'theorem [22]: $dimH^q(M,R) = dim(Ker(\triangle^q))$, which by the above construction is clearly equal to $dim(Ker(L^{\psi,q}))$; from (6.11) we conclude that

$$dimH^q(M,R) = dim(Ker\triangle^{\psi,q}). \tag{6.12}$$

Now consider the semigroups in $L^{2,q,\rho}$ with infinitesimal generators given by the self-adjoint extensions in $C^\infty\Lambda^q(M)$ of $\frac{1}{2}\triangle^{\psi q} = \frac{1}{2}\triangle^q + L_{\text{grad ln}\,\psi}$; we shall denote these semigroups as $P^q_\tau, q = 0,\ldots,n$. Clearly, P^0_τ is the stochastic process with infinitesimal generator given by the Friedrichs extension of $H(g,\psi)$. Now we note that since d commutes with $\triangle$ and Lie derivation, we conclude that d commutes with $\frac{1}{2}\triangle^{\psi q}$; consequently if α is a smooth closed form, then $\triangle^{\psi,q}d\alpha = 0$; finally, observing that:

$$\frac{\partial}{\partial\tau}(P^q_\tau d\phi)\,|_{\tau=0} = \frac{1}{2}\triangle^{\psi,q}d\phi = \frac{1}{2}d\triangle^{\psi,q-1}\phi = \frac{\partial}{\partial\tau}(dP^{q-1}_\tau)\,|_{\tau=0}\ ,$$

for any smooth $(q-1)$ form ϕ, we conclude that d commutes with P^*_τ.

With the condition of the previous paragraph (we are always assuming M is compact), we shall define other exponents of the flow F_τ of (**I**.3.65) generated by $H(g,\psi)$.

For $v \in S_xM$, and for $p \in R$ define

$$\mu(v,p) = lim_{t\to\infty}\frac{1}{t}ln\,I\!E\,|TF_\tau(v)|^p,$$

where $|A|$ denotes the operator norm of the linear transformation $A : T_xM \to T_{F_\tau(x)}M$, and the **moment exponents**:

$$\mu_x(p) = lim_{t\to\infty}\frac{1}{t}ln\,I\!E\,||T_xF_\tau||^p.$$

where $I\!E(z) = \int_\Omega z(\omega)P(d\omega)$, P the Wiener measure on Ω and $z : R^n \to R$ any random function on $R^n \cong T_xM$.

We have the following properties (Prop. 5A, [15.c])

$$i)p \to \mu_x(p)$$

is a convex function;

$$ii)p \to \frac{1}{p}\mu_x(p)$$

is increasing, and finally

$$iii) -\frac{1}{p}\mu_x(-p) \le \lambda^{(1)} \le \frac{1}{p}\mu_x(p)$$

if $p > 0$.

We shall say that the flow (or, the system) arising from (**I**.3.65) generated by $H(g, \psi)$ is **moment stable** if $\mu_x(1) < 0$, for ρ a.e. $x \in M$. From (*iii*) we see that if the flow is moment stable, then it is stable.

For our study of stability, we are interested in a semigroup induced by the lift of the diffeomorphisms of M given by the flows of (**I**.3.65) to their tangent mappings.

Let $\{F_\tau, \tau \geq 0\}$ be a flow of (**I**.3.65) such that $F_\tau(.,\omega) : M \to M$ is a diffeomorphism of class C^r, for almost all $\omega \in \Omega$. Let ϕ be a 1-form with coefficients in C^{r-1}; set

$$\tilde{P}^0_\tau(\phi) = I\!E\,(\phi \circ TF_\tau).$$

With the above assumptions, we have:

$$\tilde{P}^0_\tau(\phi) = P^1_\tau(\phi), \tag{6.13}$$

for ϕ a **closed** 1-form of class C^2. This formula follows from the application of the Ito formula (see Appendix 1) to $P_{\tau-\tau'}\phi, 0 \leq \tau' \leq \tau$

Let us give topological obstructions on M for moment stability: Let M compact with $H^1(M, R) \neq 0$; given a RCW geometry determined by a smooth metric g and a C^2 wave function $\psi \geq 0$ with associated RCW Laplacian $H(g, \psi)$, then, the flow generated by $H(g, \psi)$ with unique invariant density ρ, is not moment stable.

Indeed, by hypothesis and our above considerations on the construction of the deformed de Rham complex, we have $b_1(M) = dim(H^1(M, R)) = dim(Ker\triangle^{\psi,1})$. Then, there exists a smooth 1-form ϕ such that $\triangle^{\psi,1}\phi = 0$ (or equivalently, since $< \triangle^{\psi,1}\phi, \phi >_\rho = < d\phi, d\phi >_\rho + < \delta^\psi\phi, \delta^\psi\phi >_\rho$, ϕ is closed and $\delta^\psi\phi = 0$), and $x \in M$ and $v \in S_xM$, with $\phi(v) > 0$. Since $\phi \in Ker\triangle^{\psi,1}$ iff $P^1_\tau\phi = \phi$, we have

$$0 = lim_{t\to\infty}\frac{1}{t}ln\phi(v) = lim_{t\to\infty} ln\ P^1_\tau\phi(v),$$

which by (6.13) equals to

$$lim_{t\to\infty}\frac{1}{t}ln\, I\!E\,\phi(TF_\tau(v))$$

which is smaller or equal than

$$\mu(v, 1) \leq \mu_x(1),$$

by property *iii* above, and the result is proved.

Thus, the quantum flow generated by eqt. (**I**.3.65) with ψ a positive solution of the field equation (2.7) with g a Riemannian metric on M is not moment stable whenever M admits the existance of non-trivial harmonic 1-forms on M, or what is equivalent,

the operator $\triangle^{\psi,1}$ has a non-trivial kernel. We remark that the former description is purely topological, while the latter incoporates the RCW geometry explicitly.

We recall from **I** that the existance of a non-null harmonic Bohm-Aharonov potential introduced in **I** breaks reversibility of the quantum flow, and this condition is sufficient for the moment unstability of the flow generated by $H(g,\psi)$ as if the Bohm-Aharonov would not be present at all!

In the general case of the diffusions (**I**.3.65) generated by $H(g,\psi,A_1,A_2)$, by essentially the same argument, these flows are not moment stable provided there exists an harmonic one-form ϕ such that it preserves the full drift vector field $b = \boldsymbol{grad\ ln}\ \psi + \hat{A}_1 + \hat{A}_2$, i.e.:

$$L_b\phi = 0,$$

or, equivalently,

$$d(i_b\phi) = d(g(b,\hat{\phi})) = 0,$$

since ϕ is closed (we recall that $\hat{\phi}$ is the conjugate vector field to ϕ).

We recall that from the non-linear H-theorem presented in **I**, we concluded that b produces a local arrow of time, regulating the local fluctuation-dissipation of the system. Furthermore, from our discussion of the Lyapunov exponents we can conclude that the irrreversibile electromagnetic terms of the drift trace-torsion vector field b, are automatically canceled in their contribution to the stability of the system, since they are divergenceless (cf. (5.2)). Yet, the fact that there is a non-trivial topology on M, say a non-null Aharonov-Bohm (AB) potential which breaks detailed balance, the flow is not moment stable, even if we look exclusively to the contribution of the noise terms Y, as if the AB potential would not contribute to the drift. This points to the singularities of the AB potential, as the origin of the instabilities of the system.

VII. Conclusions

We have presented a geometrization of R.Q.M. together with a quantization of gravitation in terms of the R.C.W. geometries with torsion of the most general Weyl trace-torsion one-form. In the case of the exact component, a quantization of gravitation has been proposed by Santilli [43], applying the iso-Poincaré group structure, with isotopic unit defined by $\psi^2 I$. Yet, due to the fact that the isotopic Hilbert space action has not been manifested in the inner product (1.1), it does not preserve hermiticity observability at all times. This problem should be lifted by the introduction of this isotopic action, and thus this theory would achieve a full isotopic setting which is still lacking. We intend to retake this problem elsewhere.

We have deduced classical gravitational motions as the most probable approximations to these quantum motions, and in particular, we have deduced from them the geodesic flow, by setting the drift to 0. This is a **probabilistic** setup that extends the historical derivation of the geodesic flow **from the field equations** by Einstein, Infeld and, Hoffmann [12]. Yet, due to the remarkable coincidence between the relativistic quantum potential (which produces the quantum correlations between the parts of the quantum system) and $1/12R(g)$, we obtained instead of the geodesic flow which corresponds to spinless non-interacting test-particles submitted to an "exterior" Cartan geometry [2][8], a deviation of it due to the torsion, which corresponds to an **interacting** system. (We remark that this result does not conflict with the principle of equivalence, which is valid for test-particles). Thus, we have obtained a geometrical basis for a quantum and a classical version of de Broglie guiding wave principle, whose validity for the gravitational field was suggested -in account of the Einstein et al derivation- by Vigier[30]. The RCW geometry determines both the quantum motions and the classical ones determined as the most probable approximations to the latter. In spite of the nowhere differentiability of the quantum motions, under weak analytical conditions, we have a point dynamics on space-time (with the addendum of the canonical shift on Wiener space) which is fully **covariant**, allowing thus to define a quantum Perron-Frobenius semigroup. Of course, for the classical most probable realization we do have a classical Perron-Frobenius semigroup, which for the case of a gauge in which $b = 0$, it is the point dynamics of the geodesic flow, which thus in our theory, has **no** primordial role, in accordance with the Einstein-Infeld-Hoffmann conception.

If we further demand that our Markovian semigroups $P_\tau, \tau \geq 0$ have adjoint semigroups which are isometric, so that they are *exact Markov semigroups*, then they can be intertwined -through non-unitary operators or through coarse graining projectors- with classical dynamical systems which are **Kolmogorov** systems [18a]; this is the relativistic content in our theory of the Misra-Prigogine-Courbage theory [18b]. In fact, in this exact case, the transition density induces a PF semigroup of operators [18a], whose relation with the quantum PF semigroup associated to the quantum flow defined by integrating the s.d.eqts.(3.62) is an open problem. Finally, we have studied the qualitative behavior of the quantum flows, and seen that their ustability is linked to the fact that space-time is topologically non-trivial so that it carries non-zero harmonic forms, say, the Aharonov-Bohm potential which participates in breaking detailed balance.

Our theory is geometrically determined; this is -in our point of view- the core idea of Stochastic Differential Geometry: **Geometry determines Probability** [15,3]. It is remarkable that both the theories of Brownian motion and Relativity, that have provided in this theory for the synthesis of Quantum Mechanics and the latter, through the RCW geometries of Stochastic Differential Geometry, were the creation of Einstein, and thus his -separate- conceptions of geometrical determinism and statistical physics for which

he strongly advocated, have been linked.

Yet there are historical antecedents of blending of geometry and probabilistic structures. The causal theory of Quantum Mechanics due to de Broglie, Bohm and Vigier [19,30] relies in a Hamilton-Jacobi theory for the ψ-field which is taken to be complex, and thus it is intimately related with Symplectic Geometry.

More recently and contemporarily [29], in the formulation of statistical mechanics far from equilibrium due to Prigogine and the Brussels-Austin groups [18b], we have a Hamiltonean function on phase-space, and the theory stems from the infinitesimal generator of the corresponding Perron-Frobenius semigroup, which is the Liouville operator $\{H,.\}$, where $\{.,.\}$ denotes the Poisson bracket defined by H. Thus, also this approach stems from Symplectic Geometry [27], albeit in a trivial way. In Prigogine's theory, probability densities on phase-space satisfy an evolution equation with infinitesimal generator given by the Liouville operator instead of $H(g,\psi,A_1,A_2)$, i.e. a substitution of phase space instead of space-time, the first supported with a trivial infinitesimal generator.

A similar thesis to the one of the present article, that **quantization is geometry** has been presented for symplectic structures augmented by metrics on phase-space [32].

Our theory instead, has stemmed from Cartan geometries, yet quite remarkably, these geometries provide for a more general and natural setting for Symplectic Geometry [2,28] that the usual setting introduced from the flat canonical 1-form of classical mechanics [27]: the canonically one-form of Classical Mechanics is given by the soldering one-from on the bundle of linear frames. An interesting problem appears to be the relations between our theory and a non-trivial Hamiltonean setting provided by the substitution of the canonical symplectic form by the soldering form defined on the orthogonal bundle.

Appendix 1

The Ito formula [15]:

Suppose $\{F_\tau : \tau \geq 0\}$ is a smooth flow of $(bfI.3.65)$. Then, if $x_\tau = F_\tau(x_0)$, $v_\tau = T_{x_0}F_\tau(v_0)$, for any 1-form ϕ of class C^2, we have

$$\phi(v_\tau) = \phi(v_0) + \int_0^\tau \nabla^g\phi(B(x_s)dW_s)v_s + \int_0^\tau \phi(\nabla^g B(v_s)dW_s)$$
$$+ \frac{1}{2}\int_0^\tau trd\phi(B(x_s)-, \nabla^g B(v_\tau)-)ds + \frac{1}{2}\int_0^\tau \triangle^{\psi,1}\phi(v_s)ds, \text{(A.1)}.$$

Note that $\nabla^g\phi : TM \to T^*M$ for ϕ a 1-form on M, and $\nabla^g Y : TM \to TM$, Y a vector field on M. Since $B(x_s)dW_s$, $s \geq 0$, is a vector field on M, then so is $\nabla^g B(x_s)dW_s$, and then the first and second terms in $(A.1)$ are meaningful.

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